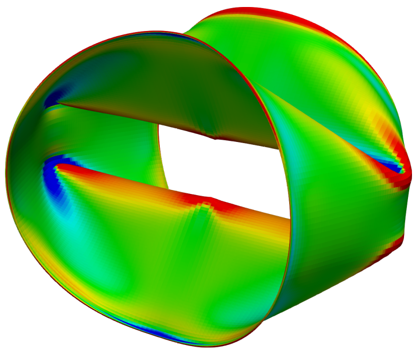
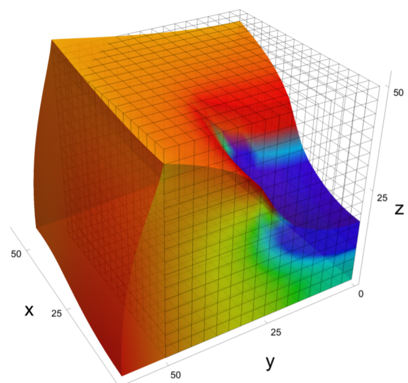
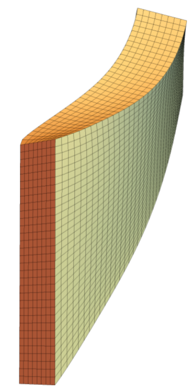
Reliable Simulation Techniques in Solid Mechanics

Development of Non-standard Discretization Methods, Mechanical and Mathematical Analysis

J. Schröder, N. Viebahn & M. Igelbüscher





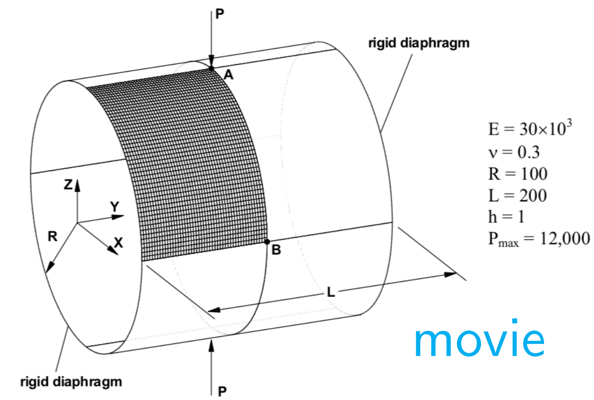


15. Deutsches LS-DYNA Forum, 15.-17. Oktober 2018, Bamberg

- Challenges in discretization techniques in solid mechanics
- Novel mixed Finite-Elements for the large deformation framework
- Least-Squares FEM - a unifying discretization technique?
- A novel Kirchhoff-Love shell formulation

Challenges in Discretization Techniques in Solid Mechanics

Displacements based low order Finite Element formulations tend to behave suspiciously stiff in various situations (e.g. incompressibility, bending dominated problems, anisotropy, thin structures)..



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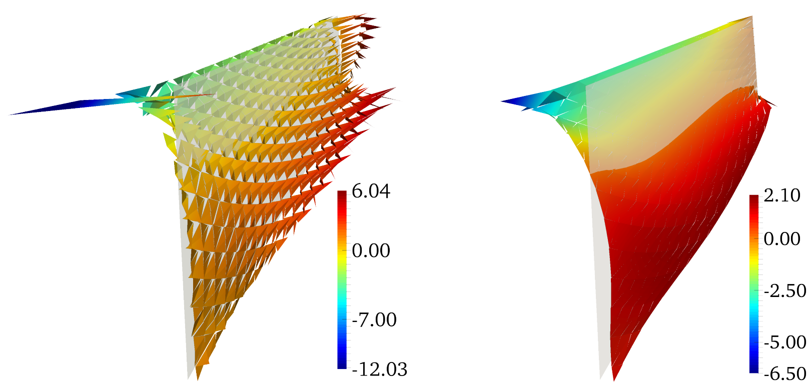
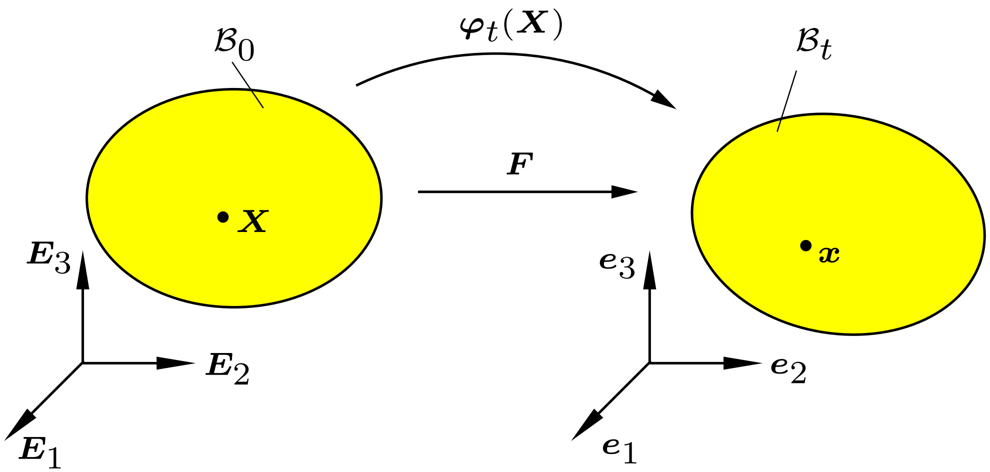
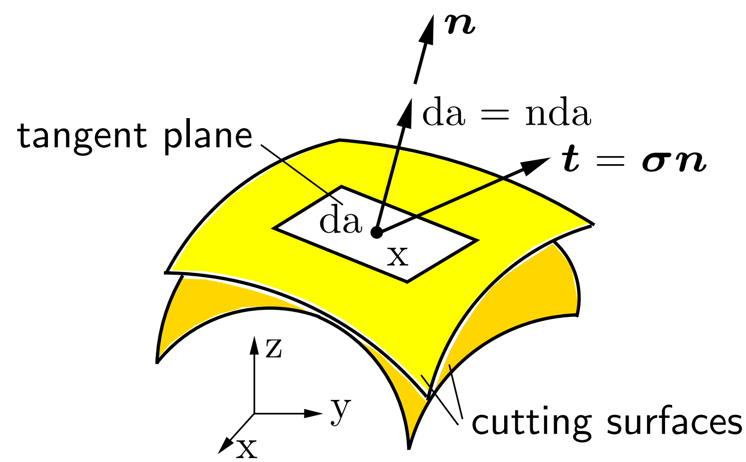


.. and their stress approximation suffers due to oscillations, especially in the incompressible regime.

Non-standard discretization methods may improve the results tremendously.

Kinematics; Deformation and Stress Measures





Deformation gradient

$$\boldsymbol{F}(\boldsymbol{X}) := \text{Grad}\boldsymbol{\varphi}_t(\boldsymbol{X}) = \text{Grad}\boldsymbol{x}$$

Right & left Cauchy-Green tensor; Green-Lagrange strain tensor

$$\boldsymbol{C} := \boldsymbol{F}^T\boldsymbol{F}\ ; \quad \boldsymbol{b} = \boldsymbol{F}\boldsymbol{F}^T\ ; \quad \boldsymbol{E} := \tfrac{1}{2}(\boldsymbol{C} - \boldsymbol{1})\ ; \quad \text{Lin}[\boldsymbol{E}] =: \boldsymbol{\varepsilon}$$

Piola transformation ($\boldsymbol{\sigma}$ - Cauchy stresses, $\boldsymbol{P}$ - 1^{st} Piola-Kirchhoff stresses)

$$\boldsymbol{t}\,da = \boldsymbol{t}_0\,dA \ : \quad \boldsymbol{\sigma n}\,da = \boldsymbol{\sigma}\text{Cof}\boldsymbol{F}\,d\boldsymbol{A} = \boldsymbol{P}\,d\boldsymbol{A} \ \rightarrow \ \boldsymbol{P} = \boldsymbol{\sigma}\text{Cof}\boldsymbol{F} = J\boldsymbol{\sigma}\boldsymbol{F}^{-T}$$

Kirchhoff stress tensor $\boldsymbol{\tau} = \text{J}\boldsymbol{\sigma}$, 2^{nd} Piola-Kirchhoff stresses $\boldsymbol{S} := \boldsymbol{F}^{-1}\boldsymbol{P}$

Some keystones in Mixed FEM for Solid Mechanics

←Hellinger [1913], Encyklopädie der math. Wissenschaften

1950

Reissner [1950]
On a variational theorem in elasticity

Washizu [1955]
On the variational principles in elasticity..

1960

Veubeke [1965]
Displacement and equilibrium models..

Zienkiewicz et al. [1971]
Reduced integration technique in general analysis of..

Babuška [1973]
The FEM with Lagrangian Multipliers

1970

Wilson [1973]
Incompatible Displacement Models

Nagtegaal et al. [1974]
On numerically accurate FE solutions in the ...

Brezzi [1974]
On the existence, uniqueness and ...

1980

Hughes [1980]
Generalization of selective integration procedures...

Pian & Sumihara [1984]
Rational approach for assumed stress finite elements

Arnold et al. [1984]
PEERS: A new mixed finite element for plane elasticity

1990

Simo & Rifai [1990]
A class of mixed assumed strain methods and ...

Pantuso & Bathe [1995]
A four-node quadrilateral mixed-interpolated element...

Wriggers & Reese [1996]
A note on enhanced strain methods for large deformations

2000

Glaser & Armero [1997]
On the formulation of enhanced strain FE in finite deformations...

Bischoff, Ramm & Braess [1999]
A class of equivalent enhanced assumed strain and hybrid stress FE

Reese, Wriggers & Reddy [2000]
A new locking-free brick element technique for large deformation...

2010

Korelc et al. [2010]
An improved EAS brick element for finite deformations

Schröder et al. [2011]
A new mixed finite element based on different approximations of the...

Auricchio et al. [2013]
Approximation of incompressible large deformation elastic ...

Mixed FEM in Solid Mechanics - a brief introduction

The terminus **Mixed** is used when different fields are introduced independently.

„Classical“ problem of Linear Elasticity:

$$\text{Find } \boldsymbol{u} \text{ such that:} \quad \text{Div}[\mathbb{C} : \nabla^s \boldsymbol{u}] + \boldsymbol{f} = \boldsymbol{0} \qquad \text{on} \quad \mathcal{B}$$

Mixed two field problem of Linear Elasticity:

$$\text{Find } (\boldsymbol{\sigma}, \boldsymbol{u}) \text{ such that:} \begin{cases} \text{Div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} & \text{on} \quad \mathcal{B} \\ \mathbb{C}^{-1} : \boldsymbol{\sigma} = \nabla^s \boldsymbol{u} & \text{on} \quad \mathcal{B} \end{cases}$$

Mixed three field problem of Linear Elasticity:

$$\text{Find } (\boldsymbol{\varepsilon}, \boldsymbol{u}, \boldsymbol{\sigma}) \text{ such that:} \begin{cases} \text{Div}\, \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} & \text{on} \quad \mathcal{B} \\ \boldsymbol{\sigma} = \mathbb{C} : \boldsymbol{\varepsilon} & \text{on} \quad \mathcal{B} \\ \boldsymbol{\varepsilon} = \nabla^s \boldsymbol{u} & \text{on} \quad \mathcal{B} \end{cases}$$

Mixed FEM in Solid Mechanics - a brief introduction

Discretization of a Mixed-Galerkin approach results in an algebraic system of the general form

$$\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}_u \\ \boldsymbol{d}_\sigma \end{bmatrix} = \begin{bmatrix} \boldsymbol{f} \\ \boldsymbol{g} \end{bmatrix}$$

This **saddle-point** structure reveals the major challenge in the construction of mixed finite elements, because **existence** and **uniqueness** of a solution cannot be guaranteed in general.

The discretization of the individual field (dofs $\boldsymbol{d}_u$ and $\boldsymbol{d}_\sigma$) have to be cautiously balanced, with regard of the conditions of well-posedness for mixed FE by Babuška [1973] and Brezzi [1974].

However, the immediate calculation of the field of interests (e.g. stresses, pressure, ..) often worth the additional efforts.

Assumed Stress Elements in Linear Elasticity

The solution of the elasticity problem with body $\mathcal{B} \in \mathbb{R}^3$, with $\boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla^s \boldsymbol{u}$

$$
\begin{aligned}
&\mathrm{div}\boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0} && \text{on } \mathcal{B}\\
&\mathbb{C}^{-1} : \boldsymbol{\sigma} = \boldsymbol{\varepsilon}(\boldsymbol{u}) && \text{on } \mathcal{B}\\
&\boldsymbol{u} = \boldsymbol{0} && \text{on } \partial\mathcal{B}_u\\
&\boldsymbol{\sigma}\boldsymbol{n} = \bar{\boldsymbol{t}} && \text{on } \partial\mathcal{B}_\sigma
\end{aligned}
$$

is equivalent to the Hellinger-Reissner principle (satisfying the displacement boundary conditions a priori) which seeks a saddle-point $(\boldsymbol{\sigma}, \boldsymbol{u}) \in L^2(\mathcal{B}) \times H_0^1(\mathcal{B})$

$$
\Pi^{\mathrm{HR}}(\boldsymbol{\sigma}, \boldsymbol{u}) = \int_{\mathcal{B}} \left(-\frac{1}{2}\boldsymbol{\sigma} : \mathbb{C}^{-1} : \boldsymbol{\sigma} + \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\boldsymbol{u}) \right) \mathrm{d}V - \int_{\partial\mathcal{B}_\sigma} \boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A
$$

$$
\begin{aligned}
\delta_u \Pi^{\mathrm{HR}} &= \int_{\mathcal{B}} \boldsymbol{\varepsilon}(\delta\boldsymbol{u}) : \boldsymbol{\sigma} \, \mathrm{d}V - \int_{\partial\mathcal{B}_\sigma} \delta\boldsymbol{u} \cdot \bar{\boldsymbol{t}} \, \mathrm{d}A &= 0 \quad \forall\, \delta\boldsymbol{u} \in H_0^1(\mathcal{B})\\
\delta_\sigma \Pi^{\mathrm{HR}} &= \int_{\mathcal{B}} \delta\boldsymbol{\sigma} : (\boldsymbol{\varepsilon}(\boldsymbol{u}) - \mathbb{C}^{-1} : \boldsymbol{\sigma}) \mathrm{d}V &= 0 \quad \forall\, \delta\boldsymbol{\sigma} \in L^2(\mathcal{B})
\end{aligned}
$$

Discretization

The displacements and stresses defined on the isoparametric space are

$$\underline{\boldsymbol{u}} = \underline{\boldsymbol{N}}\,\underline{\boldsymbol{d}} \quad \text{and} \quad \underline{\boldsymbol{\varepsilon}} = \underline{\boldsymbol{B}}\,\underline{\boldsymbol{d}}$$

$$\underline{\widehat{\boldsymbol{\sigma}}} = (\widehat{\sigma}_{11}, \widehat{\sigma}_{22}, \widehat{\sigma}_{12})^T = \underline{\widehat{\mathbb{L}}}(\boldsymbol{\xi})\,\underline{\boldsymbol{\beta}}\,,$$

where $\underline{\boldsymbol{N}}$ contains the bilinear shape functions, $\underline{\boldsymbol{B}}$ its spatial derivatives, $\underline{\boldsymbol{d}}$ the nodal displacements, $\underline{\boldsymbol{\beta}}$ the element-wise stress unknowns and $\underline{\widehat{\mathbb{L}}}$ the corresponding interpolation functions with the structure

$$\underline{\widehat{\mathbb{L}}} = \text{diag}(\underline{\widehat{\mathbb{L}}}_{11}, \underline{\widehat{\mathbb{L}}}_{22}, \underline{\widehat{\mathbb{L}}}_{12})\,.$$

5-parameter based interpolation, proposed by Pian & Sumihara [1984]

$$\underline{\widehat{\mathbb{L}}}_{11} = (1,\ \eta)\,, \quad \underline{\widehat{\mathbb{L}}}_{22} = (1,\ \xi)\,, \quad \underline{\widehat{\mathbb{L}}}_{12} = (1)\,.$$

Uniform convergence has been proven by Yu, Xie & Carstensen [2011]

Boundary Value Problem Hyperelasticity

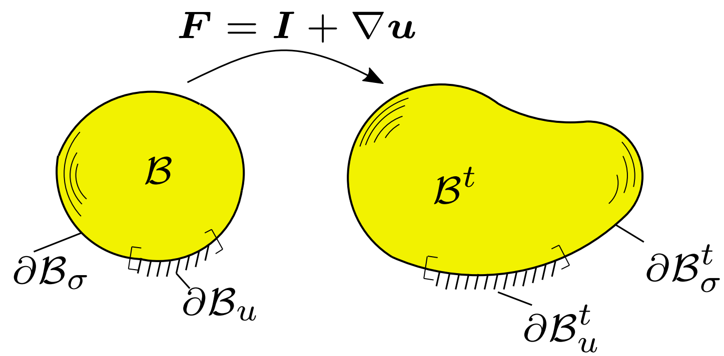
Let the second Piola Kirchhoff stress $\boldsymbol{S}$ and the displacements $\boldsymbol{u}$ be independent quantities. Then the BVP can be given with $\mathcal{B} \in \mathbb{R}^3$, $\boldsymbol{F} = \boldsymbol{I} + \nabla_X \boldsymbol{u}$, $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{C} - \boldsymbol{I})$ and $\boldsymbol{P} = \boldsymbol{F}\boldsymbol{S}$

$$\text{Div}\boldsymbol{P} + \boldsymbol{f} = \boldsymbol{0} \quad \text{on } \mathcal{B}$$

$$\frac{\partial \chi(\boldsymbol{S})}{\partial \boldsymbol{S}} = \boldsymbol{E} \quad \text{on } \mathcal{B}$$

$$\boldsymbol{u} = \overline{\boldsymbol{u}} \quad \text{on } \partial\mathcal{B}_u$$

$$\boldsymbol{P}\boldsymbol{N} = \overline{\boldsymbol{t}} \quad \text{on } \partial\mathcal{B}_\sigma$$



where $\chi(\boldsymbol{S})$ is a complementary stored energy. St. Venant type nonlinear elasticity

$$\chi(\boldsymbol{S}) = \frac{1}{2}\boldsymbol{S} : \mathbb{C}^{-1} : \boldsymbol{S}.$$

Unfortunately, such explicit complementary functions only exist for special cases.

Weak Form / Linearization

Assume that $\chi(\boldsymbol{S})$ exists. The corresponding potential is given by

$$\Pi^{\mathsf{HR}}(\boldsymbol{S},\boldsymbol{u}) = \int_{\mathcal{B}} (\boldsymbol{S}:\boldsymbol{E} - \chi(\boldsymbol{S}))\,\mathrm{d}V + \Pi^{\mathsf{ext}}.$$

and the weak forms follow by

$$\begin{aligned} \delta_u \Pi &= \int_{\mathcal{B}} \delta\boldsymbol{E} : \boldsymbol{S}\,\mathrm{d}V + \delta_u \Pi^{\mathsf{ext}} &= 0 \\ \delta_S \Pi &= \int_{\mathcal{B}} \delta\boldsymbol{S} : (\boldsymbol{E} - \partial_S \chi(\boldsymbol{S}))\mathrm{d}V &= 0 \end{aligned}$$

In cases where no complementary stored energy is known, the partial derivative $\partial_S \chi(\boldsymbol{S}) := \boldsymbol{E}^{\mathsf{cons}}$ can be computed iteratively in each integration point at fixed $\boldsymbol{S}$:

$$\boldsymbol{r}(\boldsymbol{E}^{\mathsf{cons}}) = \boldsymbol{S} - \partial_{\boldsymbol{E}} \psi(\boldsymbol{E})|_{\mathbf{E}^{\mathsf{cons}}} \approx \boldsymbol{0}$$

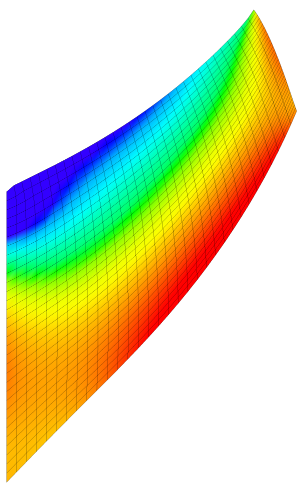
we have to update (until convergence)

$$\boldsymbol{E}^{cons} \Leftarrow \boldsymbol{E}^{cons} + \underbrace{[\partial^2_{\boldsymbol{E}\boldsymbol{E}} \psi(\boldsymbol{E})|_{\boldsymbol{E}^{cons}}]^{-1}}_{=:\,\mathbb{D}} \boldsymbol{r}(\boldsymbol{E}^{cons})$$

Cook's Membrane Problem

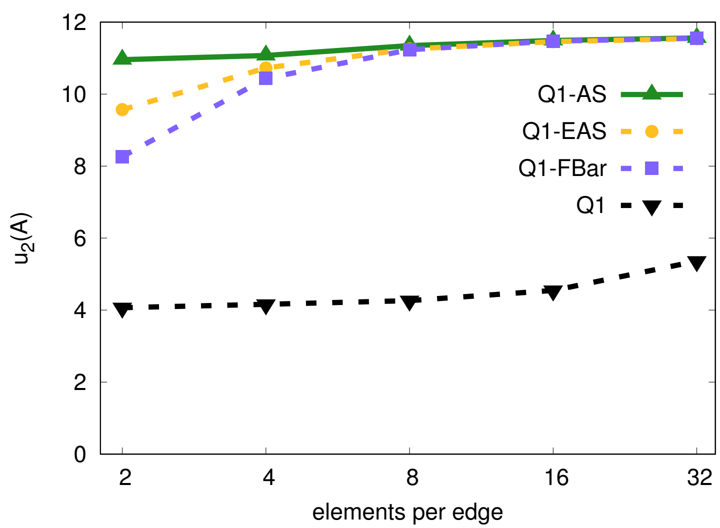
Neo-Hookean free energy: $\psi = \frac{\Lambda}{4}(J^2 - 1) - \left(\frac{\Lambda}{2} + \mu\right) \ln J + \frac{\mu}{2}(\mathrm{tr}\boldsymbol{C} - 3)$

Material parameter: $E = 200,\ \nu = 0.4999$

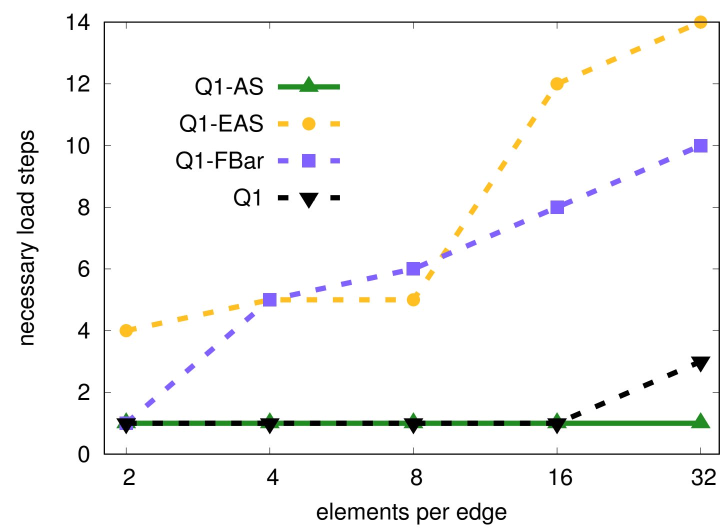


Pressure Distribution

Displacement convergence:



Necessary load steps:



Boundary Conditions:

$x = 0:$
$u_1 = 0$
$u_2 = 0$

$x = 48:$
$\bar{\boldsymbol{t}} = (0, 10)^T$

Q1-EAS: EAS Element with 4 Parameters; Simo & Rifai [1990]

Q1-FBar: Selective reduced integration technique of shape functions; Simo, Taylor, Pister [1985]

Implementation in AceGen/AceFEM

Cook's Membrane Problem

Neo-Hookean free energy: $\psi = \frac{\Lambda}{4}(J^2 - 1) - \left(\frac{\Lambda}{2} + \mu\right) \ln J + \frac{\mu}{2}(\text{tr}\boldsymbol{C} - 3)$

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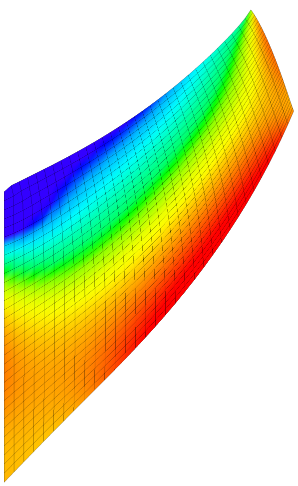
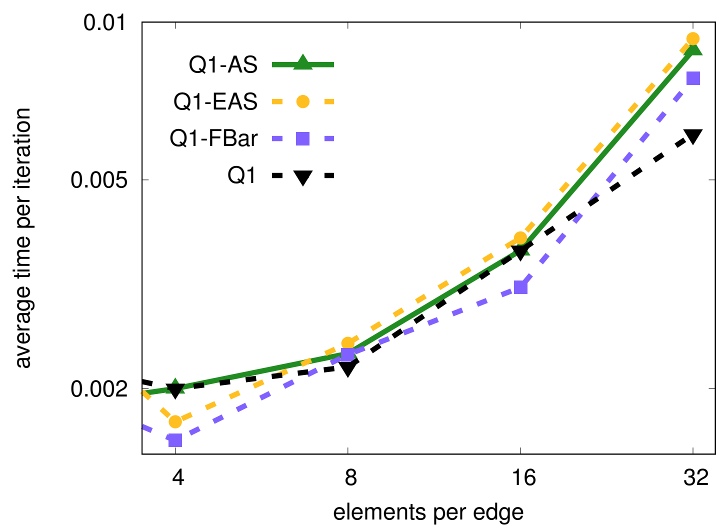
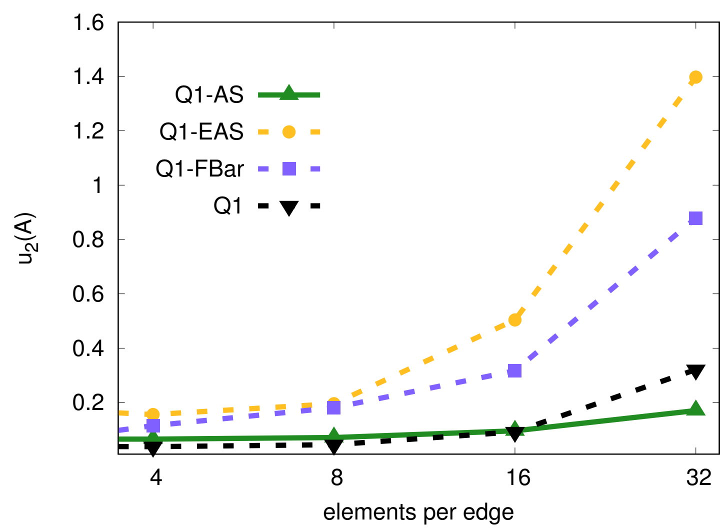


Pressure Distribution

Time per Iteration:



Total Time:



Boundary Conditions:

$x = 0:$
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$u_2 = 0$

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Q1-EAS: EAS Element with 4 Parameters; Simo & Rifai [1990]

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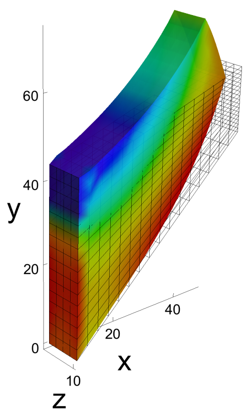
Implementation in AceGen/AceFEM

Cook's Membrane

Neo-Hookean free energy: $\psi = \frac{\Lambda}{4}(J^2 - 1) - \left(\frac{\Lambda}{2} + \mu\right) \ln J + \frac{\mu}{2}(\text{tr}\boldsymbol{C} - 3)$

Material parameter: $E = 200,\ \nu = 0.4999$

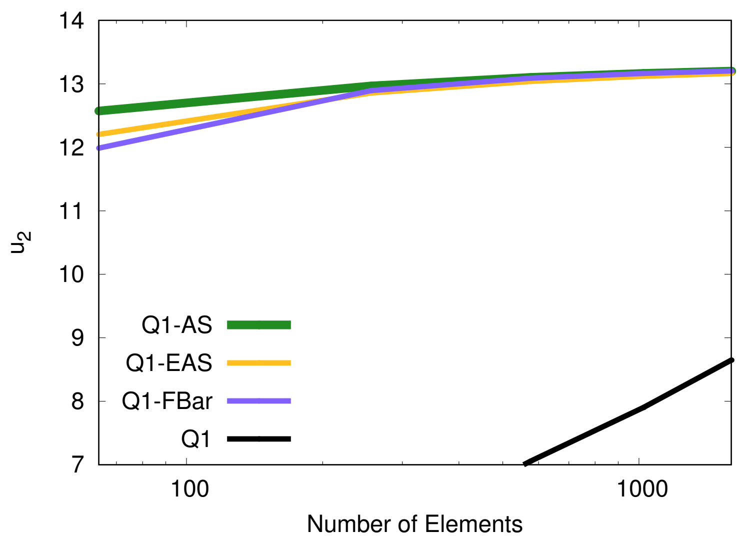


Boundary Conditions:

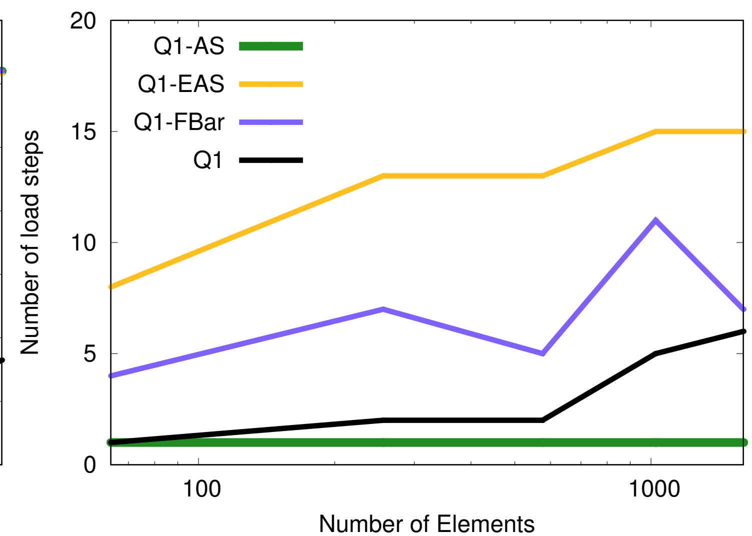
$x = 0:$
$u_1 = 0$
$u_2 = 0$
$u_3 = 0$

$x = 48:$
$\bar{\boldsymbol{t}} = (0, 10, 0)^T$

Displacement Convergence:



Necessary load steps:



Q1-EAS: EAS Element with 4 Parameters; SIMO & RIFAI [1990]

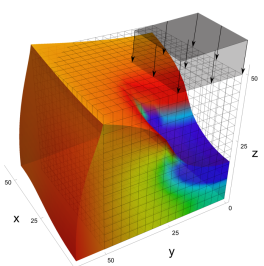
Q1-FBar: Selective reduced integration technique of shape functions; SIMO, TAYLOR, PISTER [1985]

Implementation in AceGen/AceFEM

Compression Block

Neo-Hookean free energy: $\psi = \frac{\Lambda}{4}(J^2 - 1) - \left(\frac{\Lambda}{2} + \mu\right) \ln J + \frac{\mu}{2}(\mathrm{tr}\boldsymbol{C} - 3)$

Material parameter: $E = 4.82926,\ \nu = 0.498393$



Boundary Conditions:

$Z = 0 : u_3 = 0$

$Z = 50 : u_1 = 0$
$\quad\ u_2 = 0$

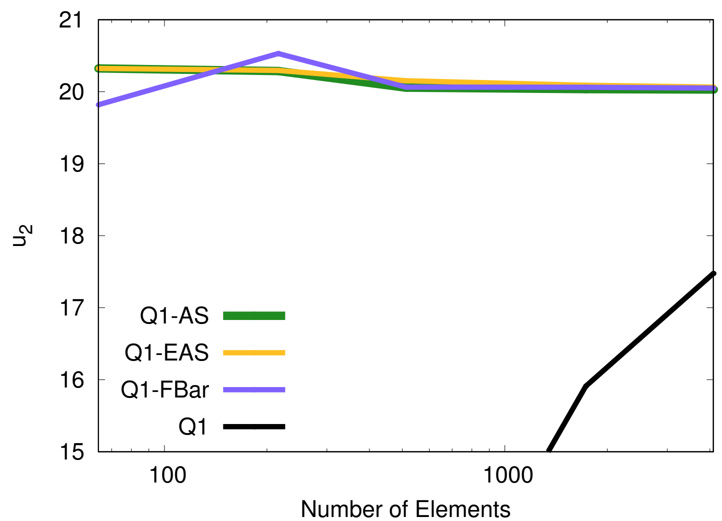
$X = 50 : u_1 = 0$

$Y = 50 : u_2 = 0$

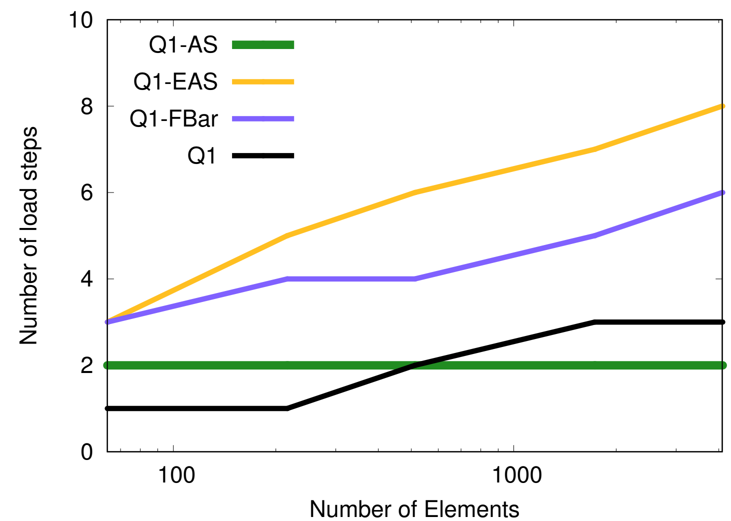
$X \leq 50 \wedge Y \leq 50 :$

$\bar{\boldsymbol{t}} = (0, 0, -3)^T$

Displacement convergence:



Necessary load steps:



Q1-EAS: EAS Element with 4 Parameters; Simo & Rifai [1990]

Q1-FBar: Selective reduced integration technique of shape functions; Simo, Taylor, Pister [1985]

Pinched Cylinder with rigid ends

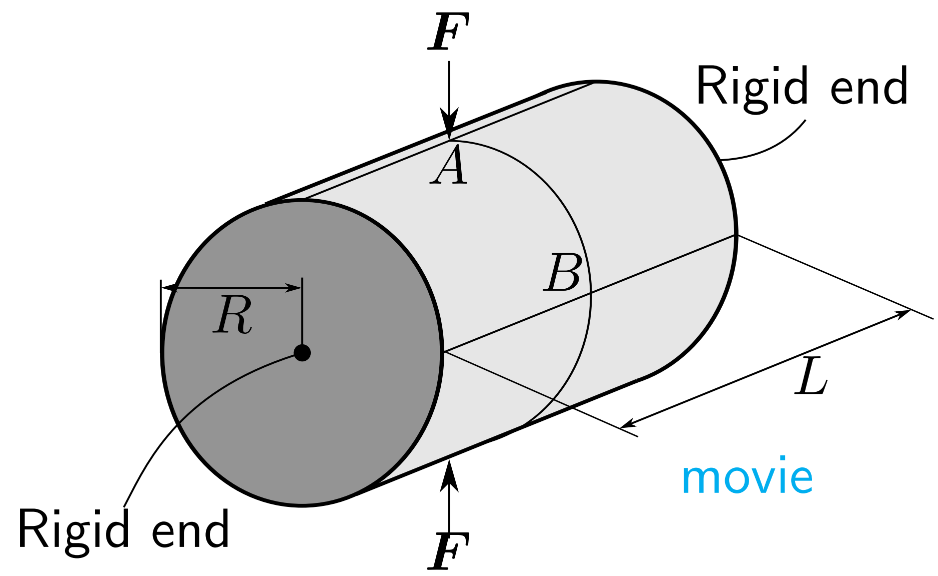
$$\psi = \frac{\Lambda}{4}(J^2 - 1) - \left(\frac{\Lambda}{2} + \mu\right) \ln J + \frac{\mu}{2}(\text{tr}\boldsymbol{C} - 3)$$

Geometrical Data: $R = 100$, $L = 200$, $h = 1$

Material Data: $E = 3 \cdot 10^4$, $\nu = 0.3$

Load: $F = 1200$

Number of elements (per height): $48 \times 48 \times 1$



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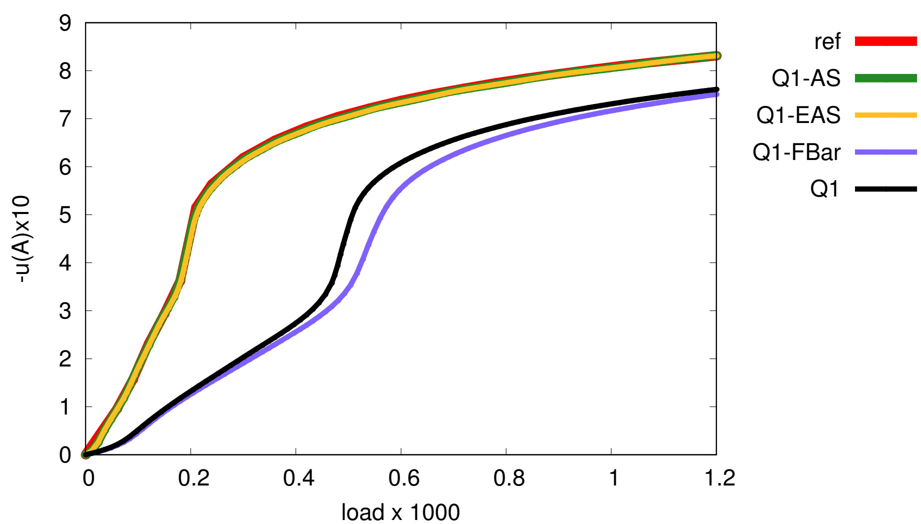
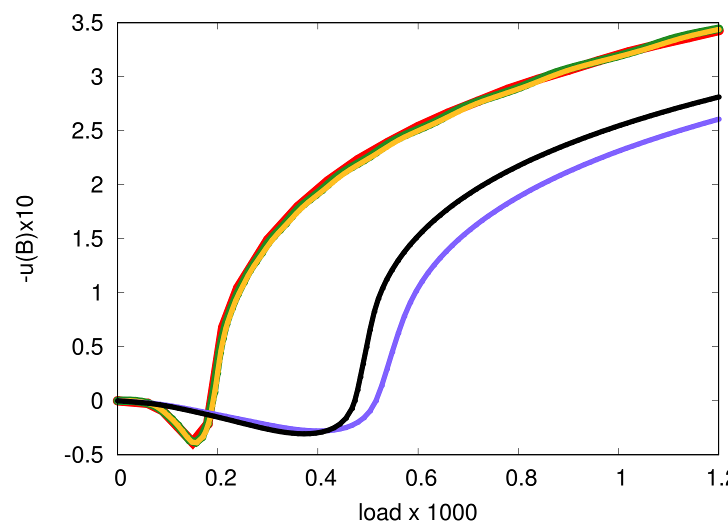




Novel Approach: SKA - Simplified Kinematics for Anisotropy

Considering an additively decoupled strain energy

$$\psi = \psi^{\text{isotropic\_part}}(\bullet) + \psi^{\text{anisotropic\_part}}(\mathcal{C})$$

where we have the following alternative for the modeling of $\psi^{\text{isotropic\_part}}$:

- Standard approximation of the deformation gradient $\boldsymbol{C}$

$$\psi^{\text{i-p}} = \psi^{\text{i-p}}(\boldsymbol{C})$$

- Volumetric-isochoric split of the free energy, $\widetilde{\boldsymbol{C}} = \widetilde{\boldsymbol{F}}^T \widetilde{\boldsymbol{F}} = J^{-2/3}\boldsymbol{C}$

$$\psi^{\text{i-p}} = \psi^{\text{vol}}(J) + \psi^{\text{unimodular}}(\widetilde{\boldsymbol{C}})$$

- Modified deformation gradient with constant volume dilatation θ

$$\psi^{\text{i-p}} = \psi(\theta^{2/3}\ \widetilde{\boldsymbol{C}})$$

→ Different approximations for θ, $\boldsymbol{C}$ and $\mathcal{C}$ can be investigated
→ The introduced kinematic-like field has to be controlled

J. Schröder, N. Viebahn, D. Balzani, P. Wriggers [2016], A novel mixed finite element for finite anisotropic elasticity; the SKA-element Simplified Kinematics for Anisotropy, CMAME [2016]

Hu-Washizu functional, Approximation of $\mathcal{C}$

$$\Pi(\boldsymbol{C},\mathcal{C},\mathcal{S}) = \int_{\mathcal{B}} \psi^{\text{i-p}}(\boldsymbol{C})\ \mathsf{d}V + \int_{\mathcal{B}} \psi^{\text{a-p}}(\mathcal{C})\ \mathsf{d}V + \int_{\mathcal{B}} \tfrac{1}{2}\mathcal{S} : (\boldsymbol{C}-\mathcal{C})\ \mathsf{d}V + \Pi^{\text{ext}}(\boldsymbol{x})$$

with $\Pi^{\text{ext}} = -\int_{\mathcal{B}} \boldsymbol{x}\cdot\boldsymbol{f}\ \mathsf{d}V - \int_{\partial\mathcal{B}} \boldsymbol{x}\cdot\boldsymbol{t}_0\ \mathsf{d}A$

$$\delta_{\boldsymbol{u}}\Pi = \int_{\mathcal{B}} \tfrac{1}{2}\,\delta\boldsymbol{C} : (\underbrace{2\,\partial_{\boldsymbol{C}}\psi^{\text{i-p}}}_{\boldsymbol{S}^{\text{i-p}}} + \mathcal{S})\ \mathsf{d}V - \int_{\mathcal{B}} \delta\boldsymbol{u}\cdot\boldsymbol{f}\ \mathsf{d}V - \int_{\partial\mathcal{B}} \delta\boldsymbol{u}\cdot\boldsymbol{t}_0\ \mathsf{d}A$$

$$\delta_{\mathcal{C}}\Pi = \int_{\mathcal{B}} \delta\mathcal{C} : (\underbrace{\partial_{\mathcal{C}}\psi^{\text{a-p}}}_{\frac{1}{2}\,\boldsymbol{S}^{\text{a-p}}} - \tfrac{1}{2}\mathcal{S})\ \mathsf{d}V = 0$$

$$\delta_{\mathcal{S}}\Pi = \int_{\mathcal{B}} \tfrac{1}{2}\,\delta\mathcal{S} : (\boldsymbol{C}-\mathcal{C})\ \mathsf{d}V = 0.$$

The identified Euler-Lagrangian equations are

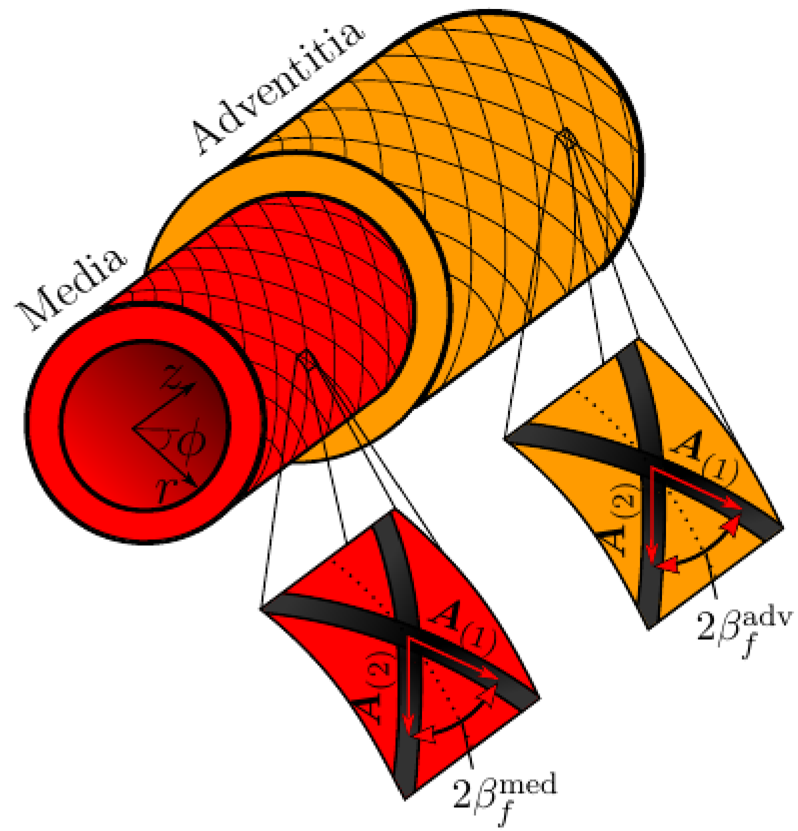
$$\text{Div}(\boldsymbol{F}\underbrace{(\boldsymbol{S}^{\text{i-p}} + \mathcal{S})}_{\boldsymbol{S}}) + \boldsymbol{f} = \boldsymbol{0}, \qquad \mathcal{S} = \boldsymbol{S}^{a-p} \qquad \text{and} \qquad \mathcal{C} = \boldsymbol{C}.$$

3D Artery - Boundary value problem

Material model (Balzani et al. [2006]):

$$\psi_{\text{i\_p}} = c_1\left(\frac{I_1}{I_3^{1/3}} - 3\right) + \varepsilon_1\left(I_3^{\epsilon_2} + I_3^{-\epsilon_2} - 2\right)$$

$$\psi_{\text{a\_p}} = \sum_{a=1}^{2} \alpha_1 \langle I_1 + J_4^{(a)} - J_5^{(a)} - 2\rangle^{\alpha_2}$$



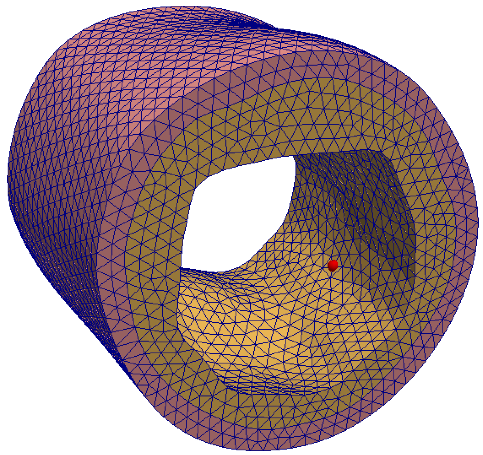
Material parameter (Brands et al. [2008]):

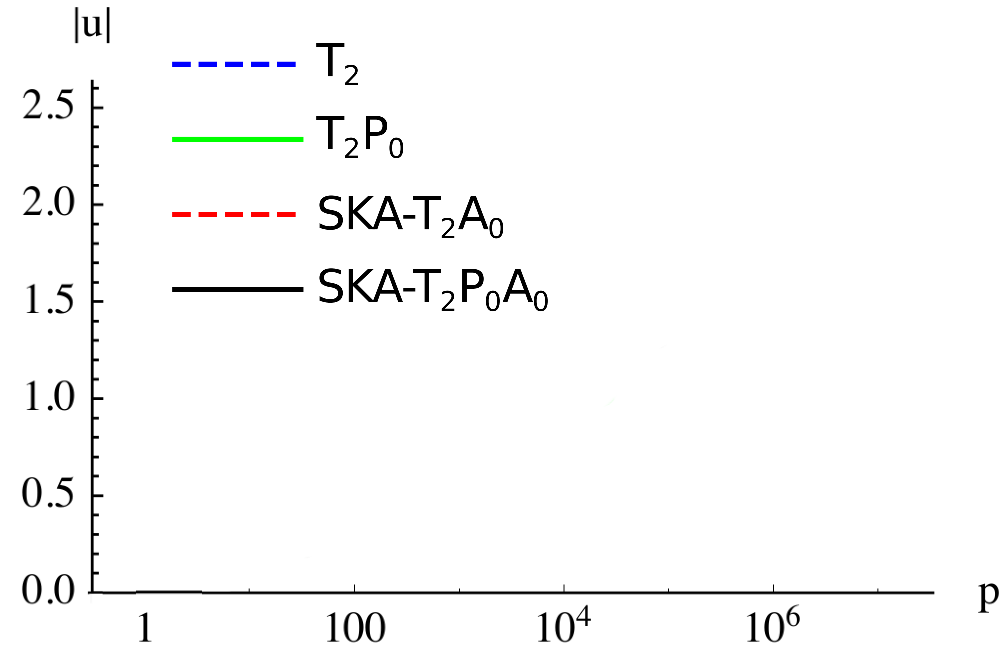
	adv.	med.
c_1	6.6	17.5
ε_1	23.9	499.8
ε_2	10.0	2.4
α_1	1503.0	30001.9
α_2	6.3	5.1
β	49.0	43.39

D. Brands, A. Klawonn, O. Rheinbach, J. Schröder [2008], Modelling and convergence in arterial wall simulations using a parallel FETI solution strategy, CMBBE, 569-583

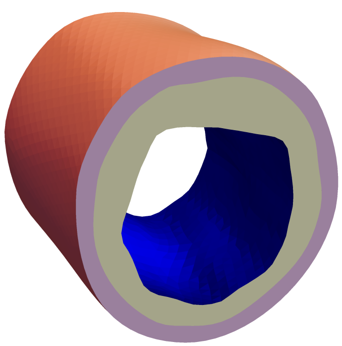
D. Balzani, P. Neff, J. Schröder, G. Holzapfel [2006] A polyconvex framework for soft biological tissues. Adjustment to experimental data, IJSS, 6052-6070

3D Artery - Supra-physical pressure - Set 2

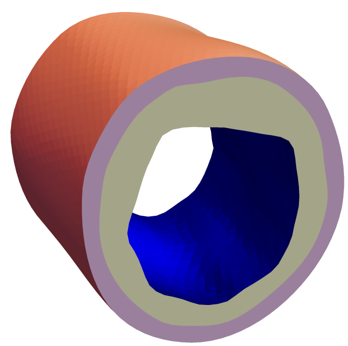




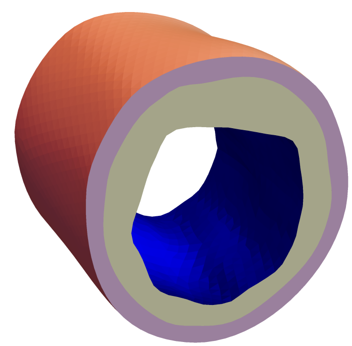
Deformed configurations for actual pressure: $p = 0$



T_2



T_2P_0

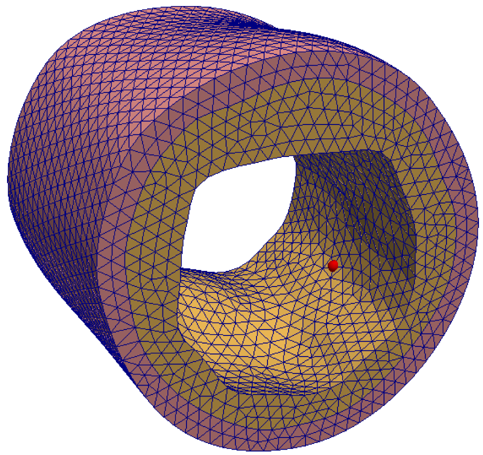


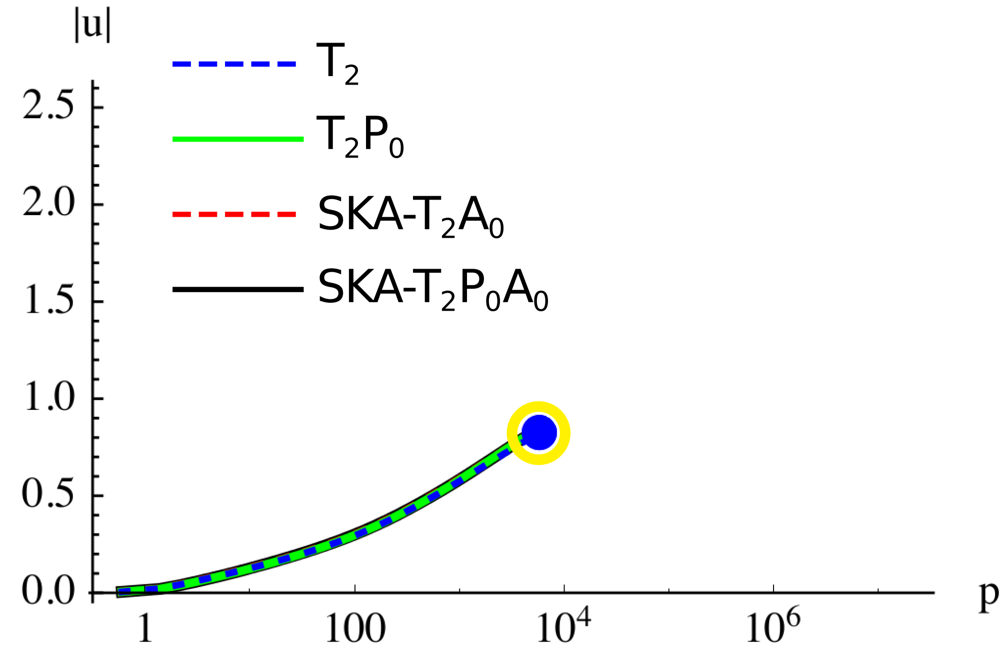
SKA-T_2A_0 / SKA-$T_2P_0A_0$

Standard formulations

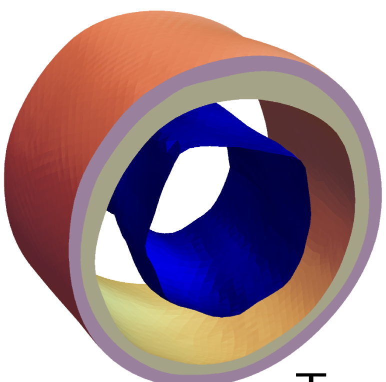
Proposed formulations

3D Artery - Supra-physical pressure - Set 2

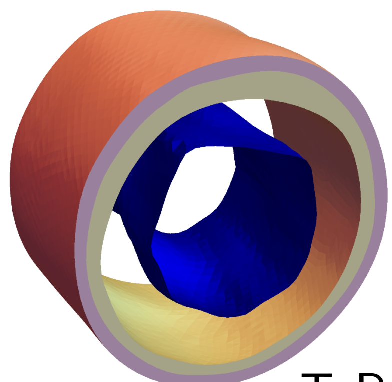




Deformed configurations for
actual pressure: $p = 5.82 \cdot 10^3$

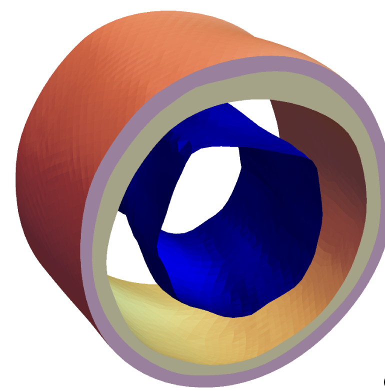


T_2



T_2P_0

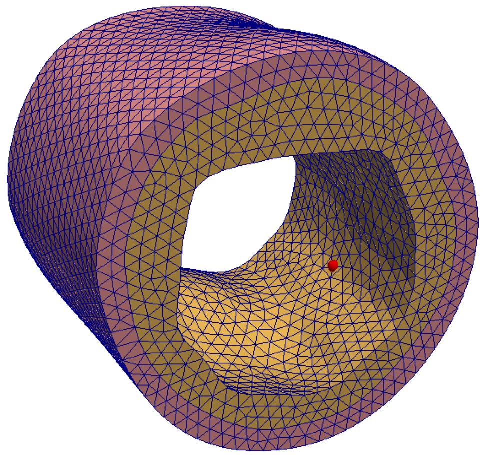
Standard formulations

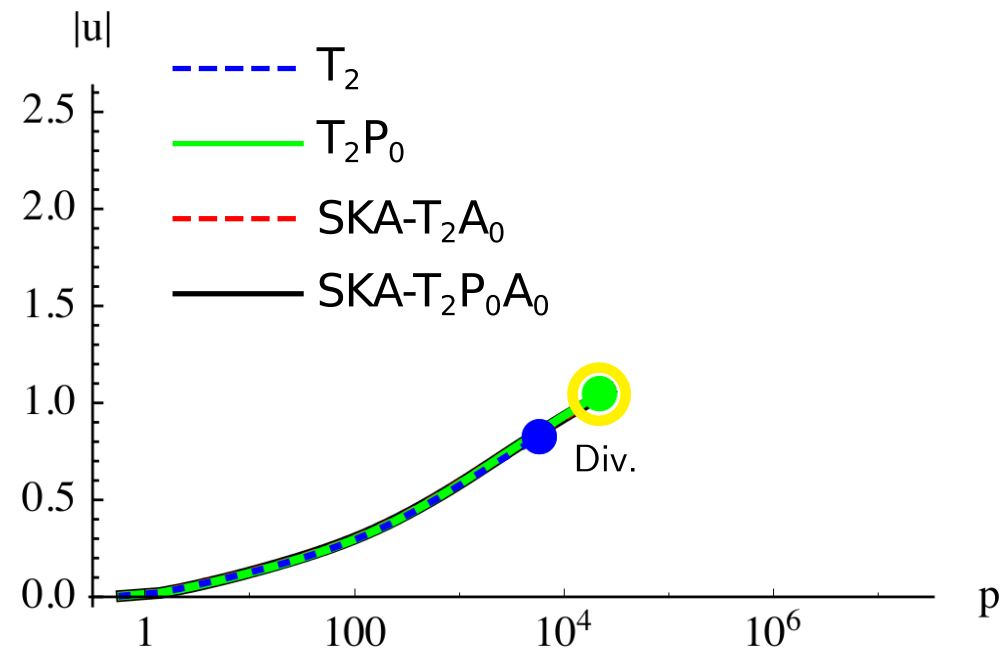


SKA-T_2A_0 /
SKA-$T_2P_0A_0$

Proposed formulations

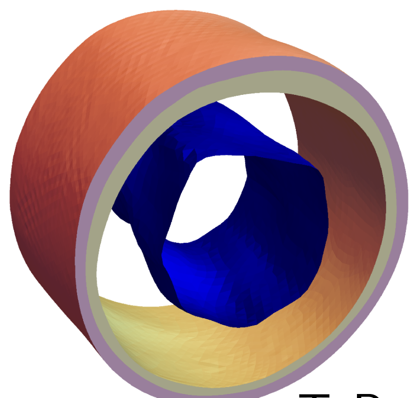
3D Artery - Supra-physical pressure - Set 2





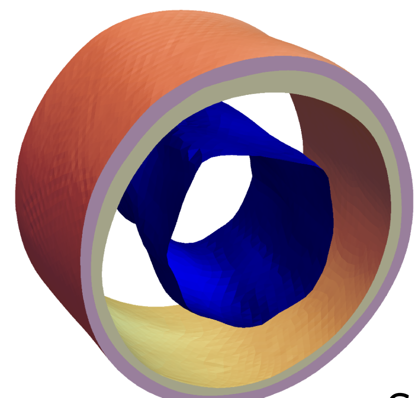
Deformed configurations for actual pressure: $p = 2.19 \cdot 10^4$

T_2



T_2P_0

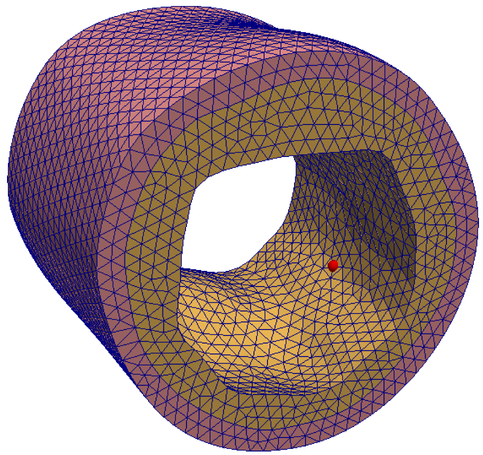
Standard formulations



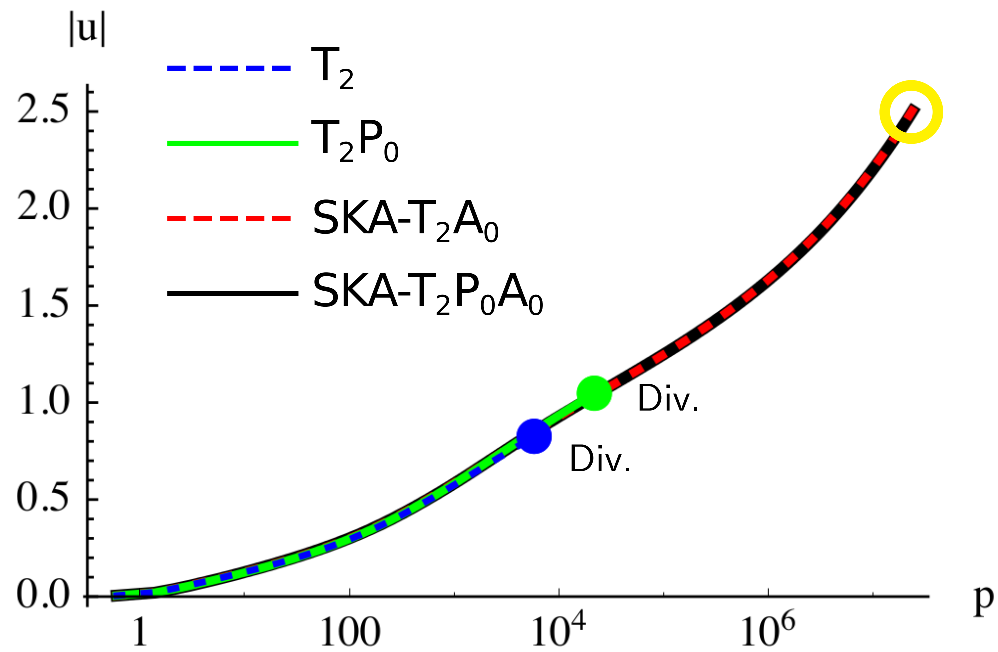
SKA-T_2A_0 / SKA-$T_2P_0A_0$

Proposed formulations

3D Artery - Supra-physical pressure - Set 2



Deformed configurations for actual pressure: $p = 10^8$



T_2

T_2P_0

Standard formulations

SKA-T_2A_0 / SKA-$T_2P_0A_0$

Proposed formulations

Motivation for Least-squares FEM

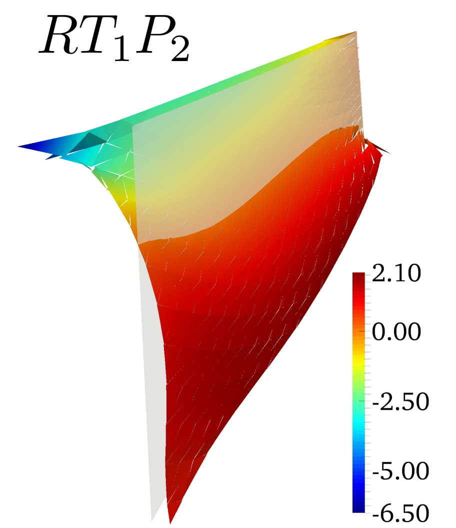
The advantage of using conform mixed $(\boldsymbol{\sigma}, \boldsymbol{u})$-based methods lies in the stress approximation, here with Raviart-Thomas functions in $H(\text{div})$, which yields continuous stress distributions in contrast to standard displacement methods (StDM).

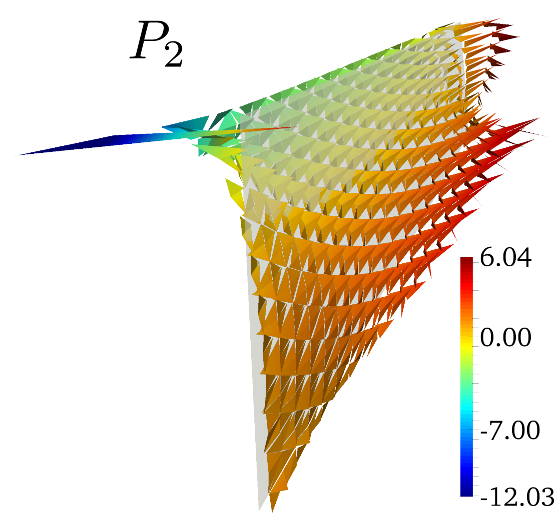
Advantages of the classical Least-Squares Method:

- LS functional leads to a minimization problem
- Not restricted by the LBB conditon
- Symmetric and positive definite matrices
- A posteriori error estimator is provided

Disadvantages of the classical Least-Squares Method:

- Lower order elements have a poor performance
- Weighting of the individual residuals is questionable

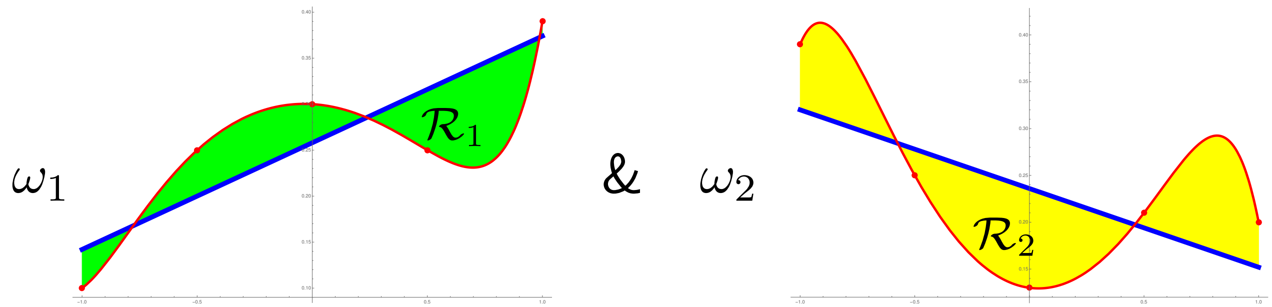




General construction of a Least-Squares Functional

To define the minimization problem, we apply the squared $L^2(\mathcal{B})$-norm to a first-order system of n differential equations, see e.g. Cai & Starke [2004],

$$\mathcal{F}(\boldsymbol{u},\boldsymbol{\sigma}) = \frac{1}{2}\Big(\|\omega_1(\operatorname{div}\boldsymbol{\sigma} + \boldsymbol{f})\|^2_{L^2(\mathcal{B})} + \|\omega_2(\boldsymbol{\sigma} - \mathbb{C} : \nabla^s \boldsymbol{u})\|^2_{L^2(\mathcal{B})}\Big) \to \text{minimize.}$$



with $\delta_{\boldsymbol{\sigma},\boldsymbol{u}}\mathcal{F} = 0$. Requirements for approximation spaces $(\boldsymbol{V}, \boldsymbol{X})$ and finite element spaces $RT_m P_k$ with

$$\boldsymbol{V} = \left\{\boldsymbol{u} \in H^1(\mathcal{B})^d\right\} \supseteq \boldsymbol{V}_h^k = \left\{\boldsymbol{u} \in H^1(\mathcal{B})^d : \boldsymbol{u}|_{\mathcal{B}_e} \in P_k(\mathcal{B}_e)^d \; \forall \, \mathcal{B}_e\right\} \; ,$$

and furthermore

$$\boldsymbol{X} = \left\{\boldsymbol{\sigma} \in H(\operatorname{div},\mathcal{B})^d\right\} \supseteq \boldsymbol{X}_h^m = \left\{\boldsymbol{\sigma} \in H(\operatorname{div},\mathcal{B})^d : \boldsymbol{\sigma}|_{\mathcal{B}_e} \in RT_m(\mathcal{B}_e)^d \; \forall \, \mathcal{B}_e\right\} \; .$$

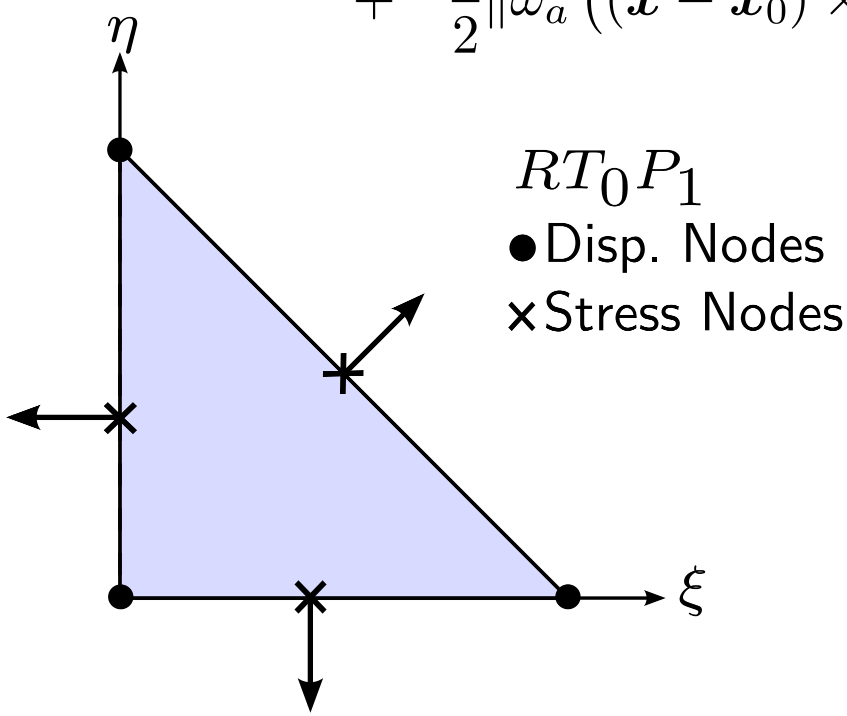
Remarks on least-squares finite element methods

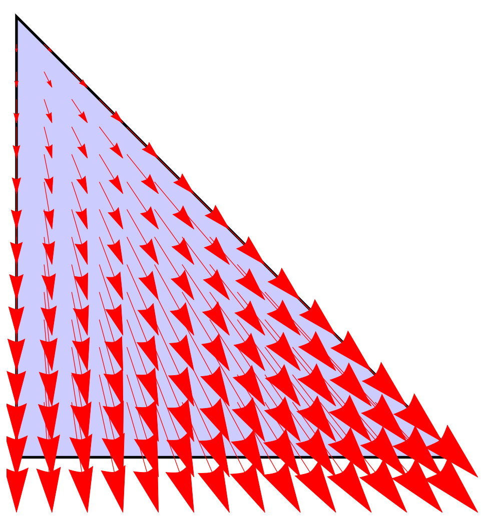
Stress-displacement LSFEM with use of Raviart-Thomas approximation functions

$$\mathcal{F}(\boldsymbol{\sigma},\boldsymbol{u}) = \frac{1}{2}\|\omega_m\,(\operatorname{div}\,\boldsymbol{\sigma}+\boldsymbol{f})\,\|^2_{L^2(\mathcal{B})} + \frac{1}{2}\|\omega_c\,(\boldsymbol{\sigma}-\mathbb{C}:\nabla^s\boldsymbol{u})\,\|^2_{L^2(\mathcal{B})}$$

and

$$\begin{aligned}\mathcal{F}(\boldsymbol{\sigma},\boldsymbol{u}) &= \frac{1}{2}\|\omega_m\,(\operatorname{div}\,\boldsymbol{\sigma}+\boldsymbol{f})\,\|^2_{L^2(\mathcal{B})} + \frac{1}{2}\|\omega_c\,(\boldsymbol{\sigma}-\mathbb{C}:\nabla^s\boldsymbol{u})\,\|^2_{L^2(\mathcal{B})}\\ &+ \frac{1}{2}\|\omega_a\,((\boldsymbol{x}-\boldsymbol{x}_0)\times(\operatorname{div}\,\boldsymbol{\sigma}+\boldsymbol{f})+\mathsf{axl}[\boldsymbol{\sigma}-\boldsymbol{\sigma}^T])\,\|^2_{L^2(\mathcal{B})}\,.\end{aligned}$$





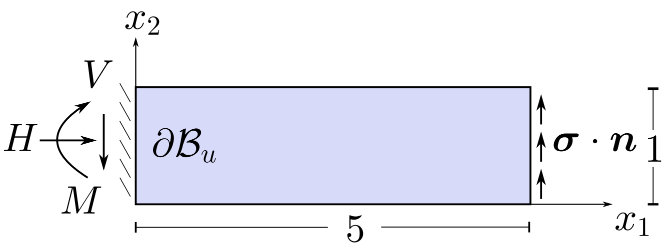
$$\psi_0^1 = \begin{pmatrix}\xi\\ \eta-1\end{pmatrix}$$

$$\psi_0^2 = \begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

$$\psi_0^3 = \begin{pmatrix}\xi-1\\ \eta\end{pmatrix}$$

RT_0P_1 dof for 2D (left) and exemplarily basis function for lower edge (right)

Approximation of reaction force for a cantilever beam



$$E = 70$$

$$\nu = 0.34$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{n} = (0, 0.1)^T$$

standard disp.:

$$H = \sum_{I\in\partial\mathcal{B}_u} F^I_{x_1}$$

$$V = \sum_{I\in\partial\mathcal{B}_u} F^I_{x_2}$$

$$M = \sum_{I\in\partial\mathcal{B}_u} F^I_{x_1}\cdot x^I_2$$

LSFEM:

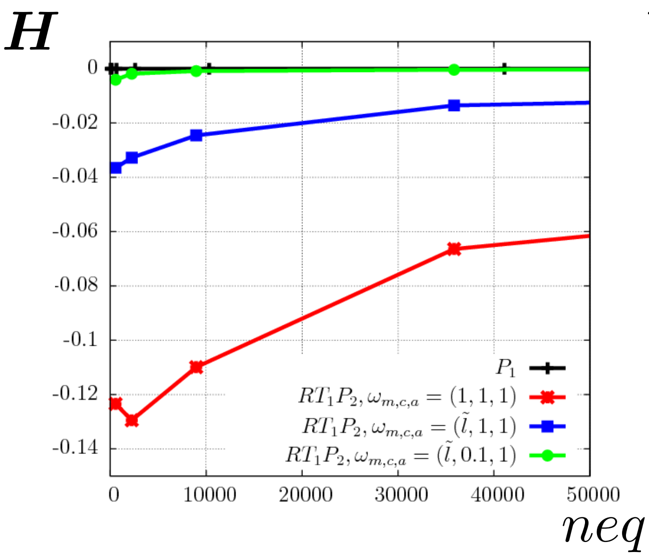
$$H = \int_{\partial\mathcal{B}_u} \sigma_{11}\,\mathrm{d}x_2$$

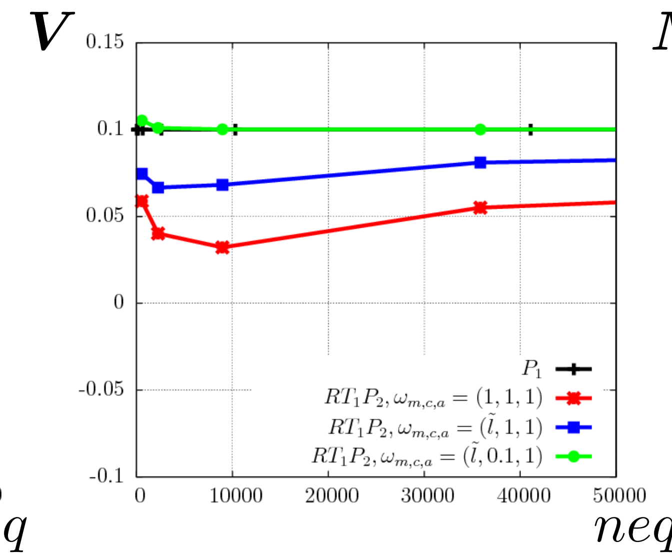
$$V = \int_{\partial\mathcal{B}_u} \sigma_{21}\,\mathrm{d}x_2$$

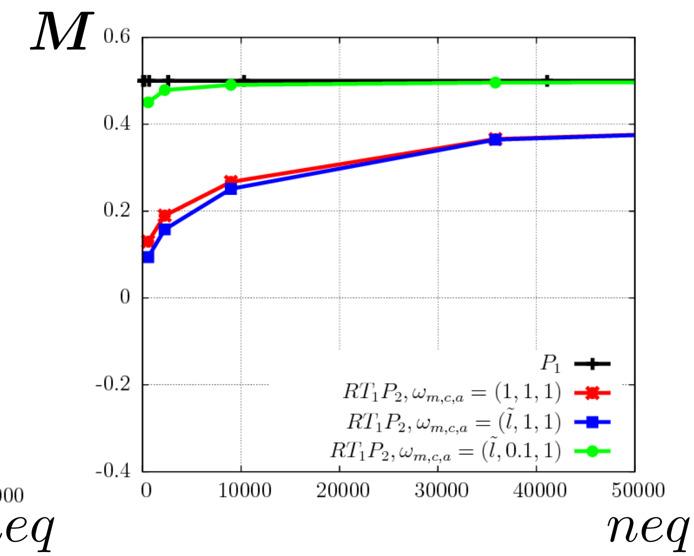
$$M = \int_{\partial\mathcal{B}_u} \sigma_{11}\cdot\hat{x}_2\,\mathrm{d}x_2$$

$$\hat{x}_2 = x_2 - x_M$$

Reaction forces compared to analytical results ($\sum H = 0, \sum V = 0.1, \sum M = 0.5$):







Least-squares functional for finite strain elasticity

Extending the formulation by adding a mathematically redundant residual cf. [3], [4], given by a stress symmetry condition, here in terms of the 2nd Piola-Kirchhoff stresses $\boldsymbol{S} = \boldsymbol{F}^{-1}\boldsymbol{P}$; $\mathcal{R}_3 = \boldsymbol{S} - \boldsymbol{S}^T$. The resulting least-squares functional yields

$$\begin{aligned} \mathcal{F} &= \frac{1}{2}\int_{\mathcal{B}} \omega_1^2 (\operatorname{Div}\boldsymbol{P} + \boldsymbol{f}) \cdot (\operatorname{Div}\boldsymbol{P} + \boldsymbol{f})\, \mathrm{d}V \\ &+ \frac{1}{2}\int_{\mathcal{B}} \omega_2^2 (\boldsymbol{P} - \rho_0\, \partial_{\boldsymbol{F}}\psi(\boldsymbol{C})) : (\boldsymbol{P} - \rho_0\, \partial_{\boldsymbol{F}}\psi(\boldsymbol{C}))\, \mathrm{d}V \\ &+ \frac{1}{2}\int_{\mathcal{B}} \omega_3^2 (\boldsymbol{F}^{-1}\boldsymbol{P} - (\boldsymbol{F}^{-1}\boldsymbol{P})^T) : (\boldsymbol{F}^{-1}\boldsymbol{P} - (\boldsymbol{F}^{-1}\boldsymbol{P})^T)\, \mathrm{d}V\,, \end{aligned}$$

based on a Neo-Hookean type free energy function $\psi(\boldsymbol{C})$ in terms of $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$

$$\psi(\boldsymbol{C}) = \frac{\mu}{2}(I_1 - 3) + \frac{\Lambda}{4}(J^2 - 1) - \left(\frac{\Lambda}{2} + \mu\right) \ln J$$

with the principal invariant $I_1 = \operatorname{tr}\boldsymbol{C}$, $J = \det \boldsymbol{F}$ and $\rho_0 = 1\,\frac{\mathrm{kg}}{\mathrm{m}^3}$.

[3] Cai & Starke [2003], SIAM J. Numer. Anal. 41:715-730

[4] Schwarz et al. [2014], Comp. Mech. 54(1):603-612

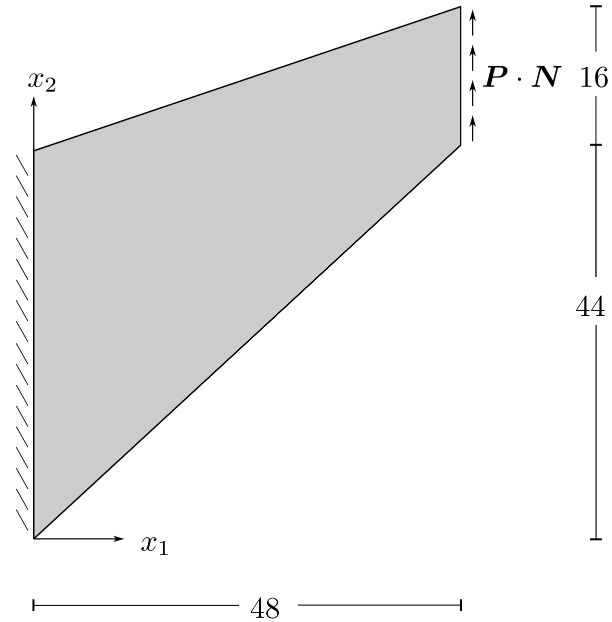
Cook's membrane problem for finite strain elasticity

Left side: $\boldsymbol{u} = (0,0)^T$

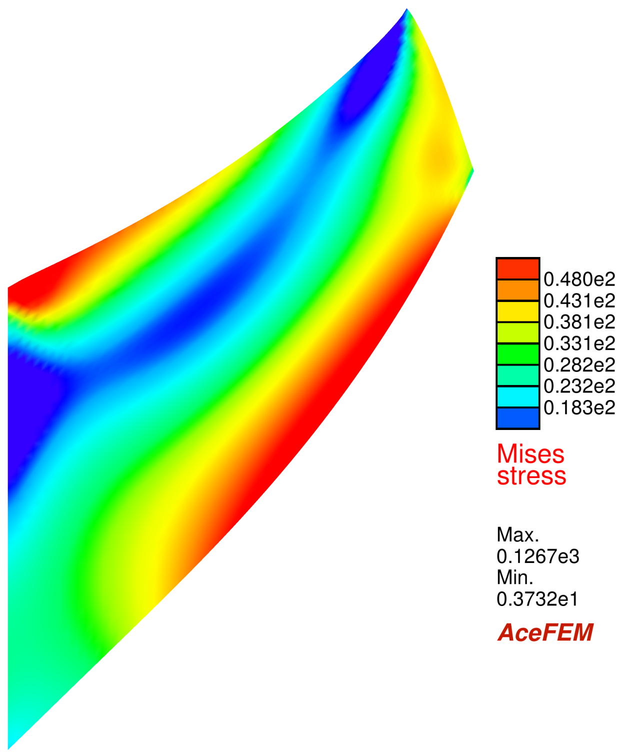
Right face: $\boldsymbol{PN} = (0,20)^T$

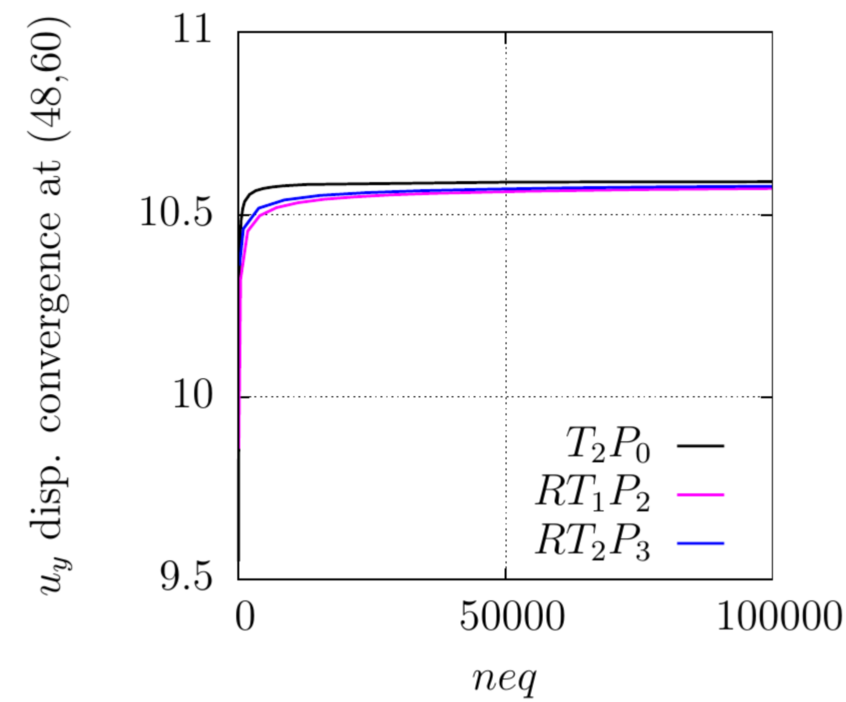
$\Lambda = 432.099,\ \nu = 185.185,$

$\omega_1 = 1$, $\omega_2 = 1/\mu$ and $\omega_3 = 10/\mu$



$\boldsymbol{\sigma}_{vM}$ distribution and convergence studie at (48,60):





Cook's membrane problem for finite strain elasticity

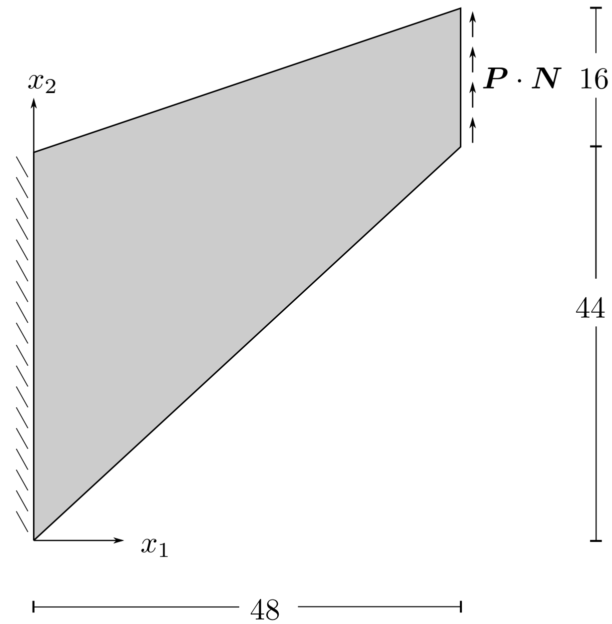
Left side: $\boldsymbol{u} = (0,0)^T$

Right face: $\boldsymbol{PN} = (0,10)^T$

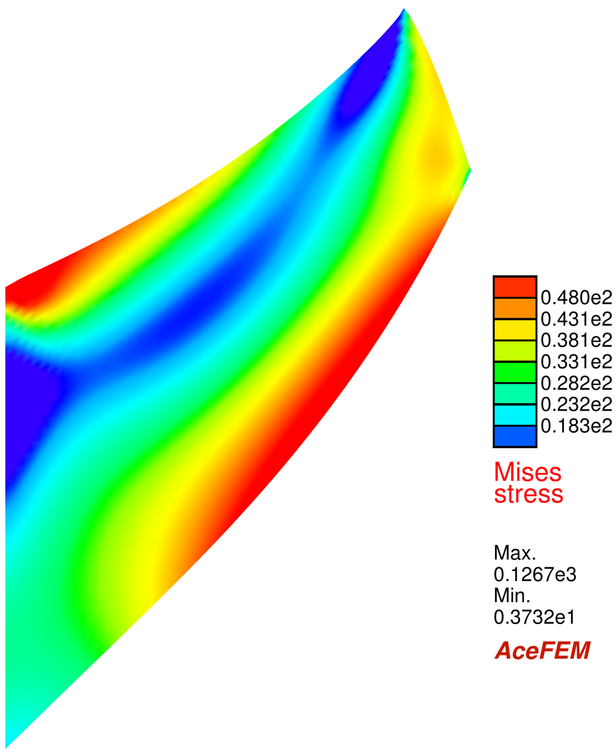
$\Lambda = (432.099, 750, 9260, 92600)$

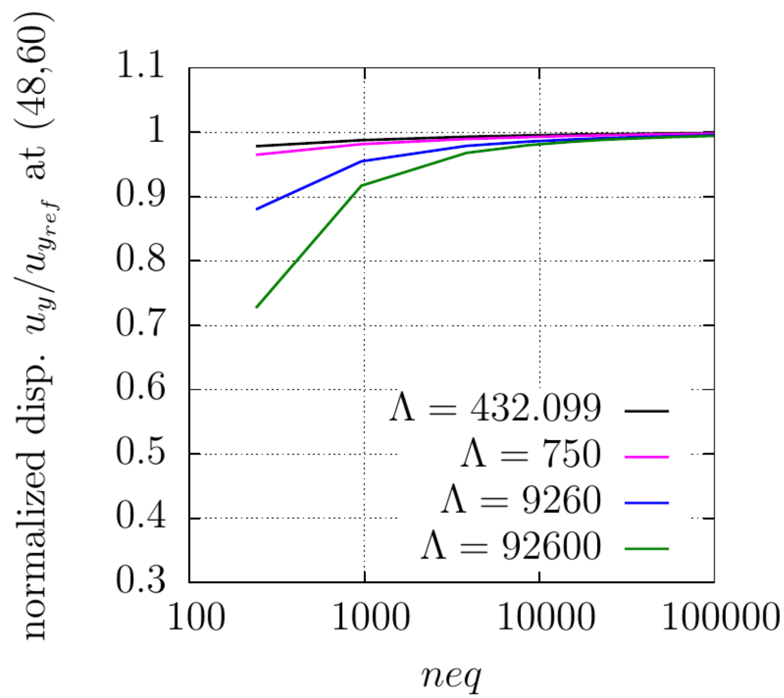
$\nu = (0.35, 0.40099, 0.490197, 0.499002)$

$\omega_1 = 1$, $\omega_2 = 1/\mu$ and $\omega_3 = 10/\mu$



$\boldsymbol{\sigma}_{vM}$ distribution and locking behavior for RT_2P_3:

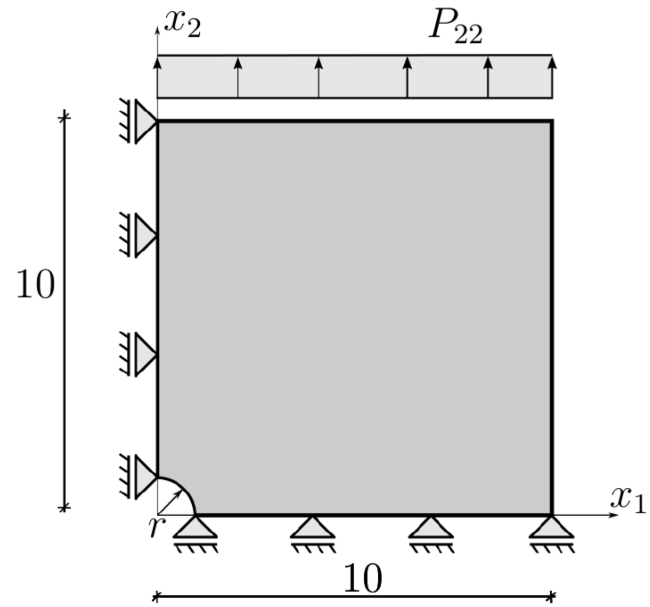




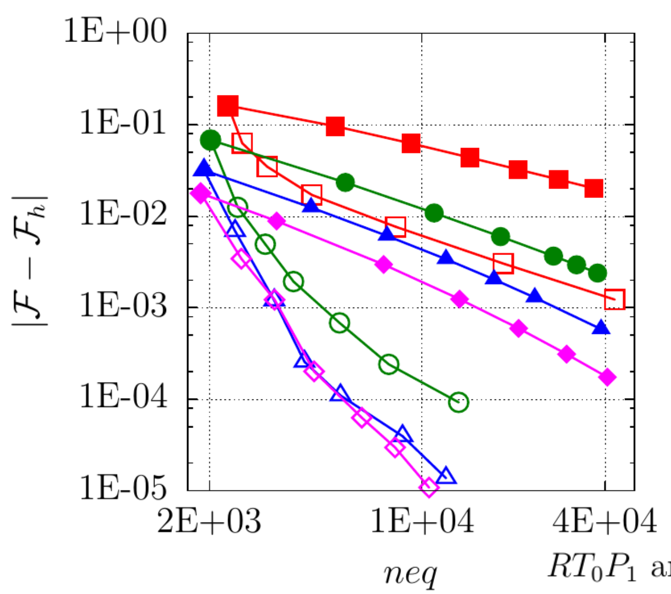
Perforated plate example for finite strain elasticity

Boundary conditions, material properties and system:

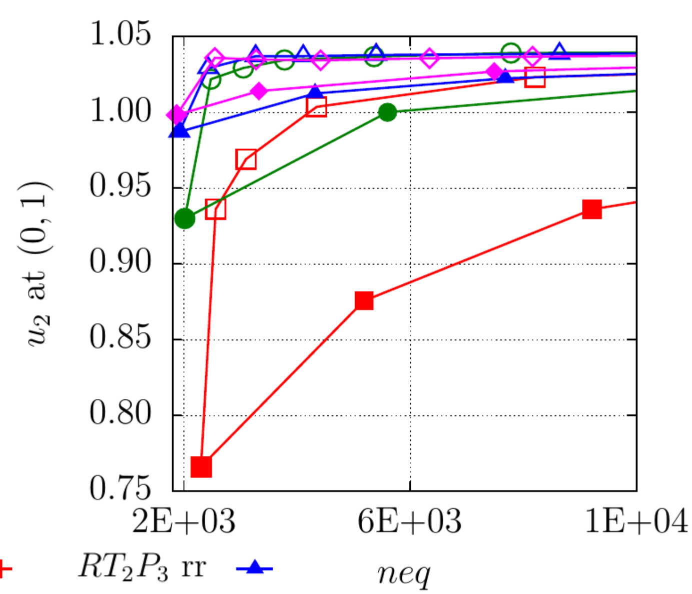
- Left side $u_1 = 0,\ P_{21} = 0$
- Lower side $u_2 = 0,\ P_{12} = 0$
- Right side $\boldsymbol{PN} = (0,0)^T$
- Upper side $\boldsymbol{PN} = (0,50)^T$
- $E = 200,\ \nu = 0.35,\ \omega_i = 1, 1/\mu, 1/\mu$



Convergence of $|\mathcal{F} - \mathcal{F}_h|$, order of convergence and u_2-displacement at (0,1):



	regular (rr)	adaptive (arD)
RT_0P_1	0.83355	1.07582
RT_1P_2	1.28705	1.80055
RT_2P_3	1.58372	3.21487
RT_3P_4	1.87417	3.92780

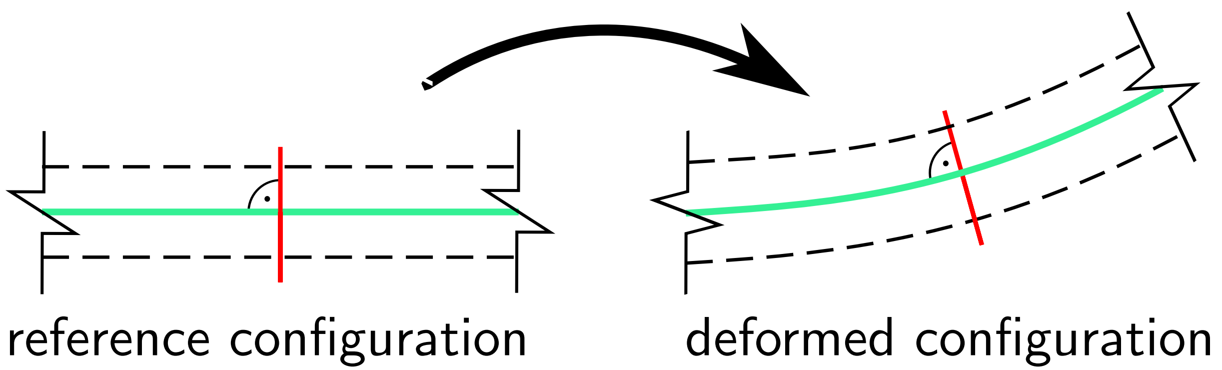


A simple triangular finite element for nonlinear thin shells - Statics, Dynamics and anisotropy

Acknowledgement: Paulo Pimenta

Based on the Kirchhoff-Love theory of plates, LOVE [1888].

Kinematic assumption: **A straight normal of the reference mid-surface remains a straight normal of the deformed mid-surface.**



Plane-stress and shear-rigid assumptions lead to a stress tensor, which is non-trivial only for the mid-plane of the shell, i.e. $\tau_{3i} = \tau_{i3} = 0$, whereas $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ span the mid-plane of the shell.

Assumptions are valid for "thin shells" with $h/L < 1/10$.

Kinematics

Based on PIMENTA, NETO, CAMPELLO [2010] and using the assumption of initial flat reference elements.

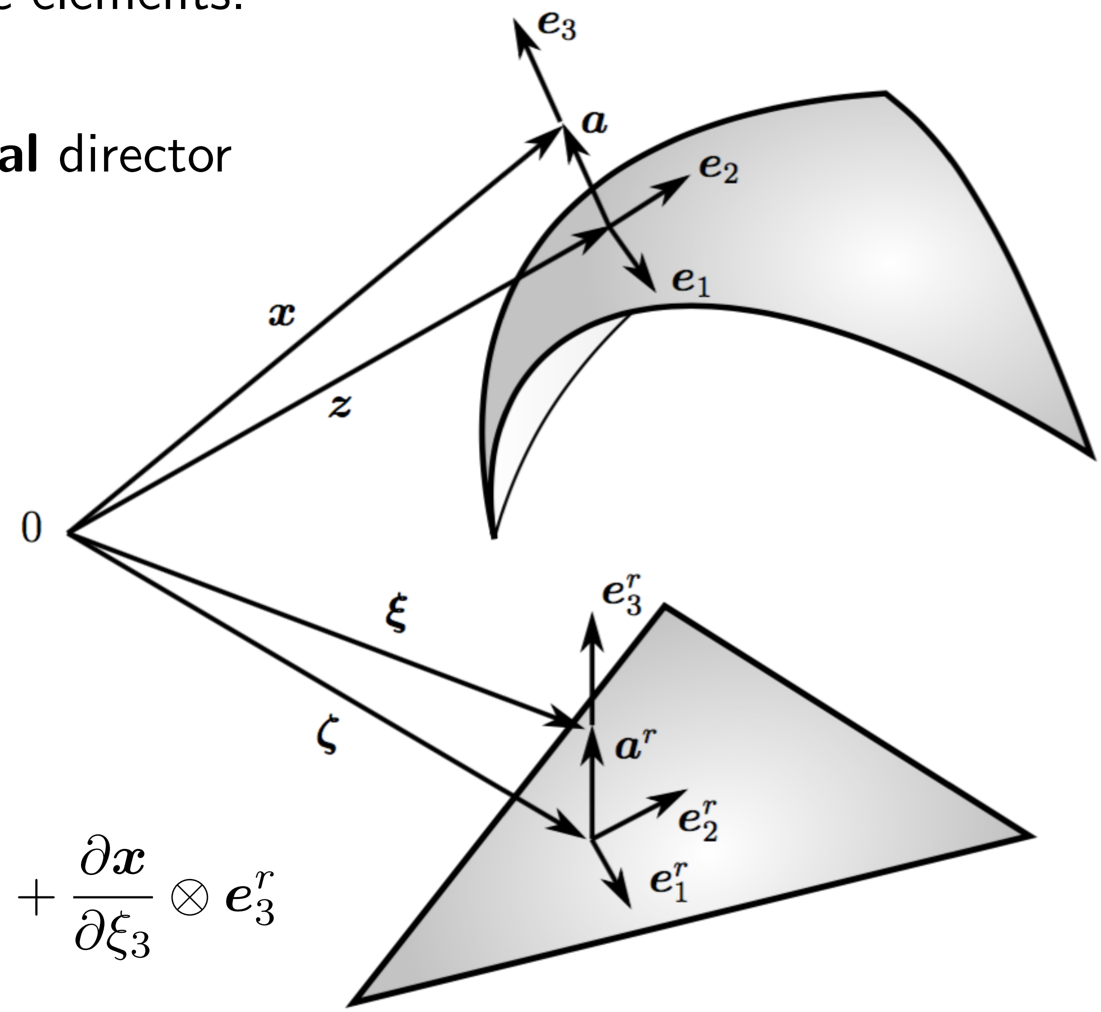
Description of material point:
Point on middle Surface + **orthogonal** director

Reference configuration: $\boldsymbol{\xi} = \boldsymbol{\zeta} + \boldsymbol{a}^r$
with $\boldsymbol{\zeta} = \xi_\alpha \boldsymbol{e}_\alpha^r$ and $\boldsymbol{a}^r = \xi_3 \boldsymbol{e}_3^r$

Current configuration: $\boldsymbol{x} = \boldsymbol{z} + \boldsymbol{a}$
with $\boldsymbol{z} = \boldsymbol{u} - \boldsymbol{\zeta}$

Orthogonal director: $\boldsymbol{a} = \boldsymbol{Q}\boldsymbol{a}^r$
with rotation tensor $\boldsymbol{Q} = \boldsymbol{e}_i \otimes \boldsymbol{e}_i^r$

Deformation gradient: $\boldsymbol{F} = \dfrac{\partial \boldsymbol{x}}{\partial \xi_\alpha} \otimes \boldsymbol{e}_\alpha^r + \dfrac{\partial \boldsymbol{x}}{\partial \xi_3} \otimes \boldsymbol{e}_3^r$



Enforcement of the C^1-Continuity

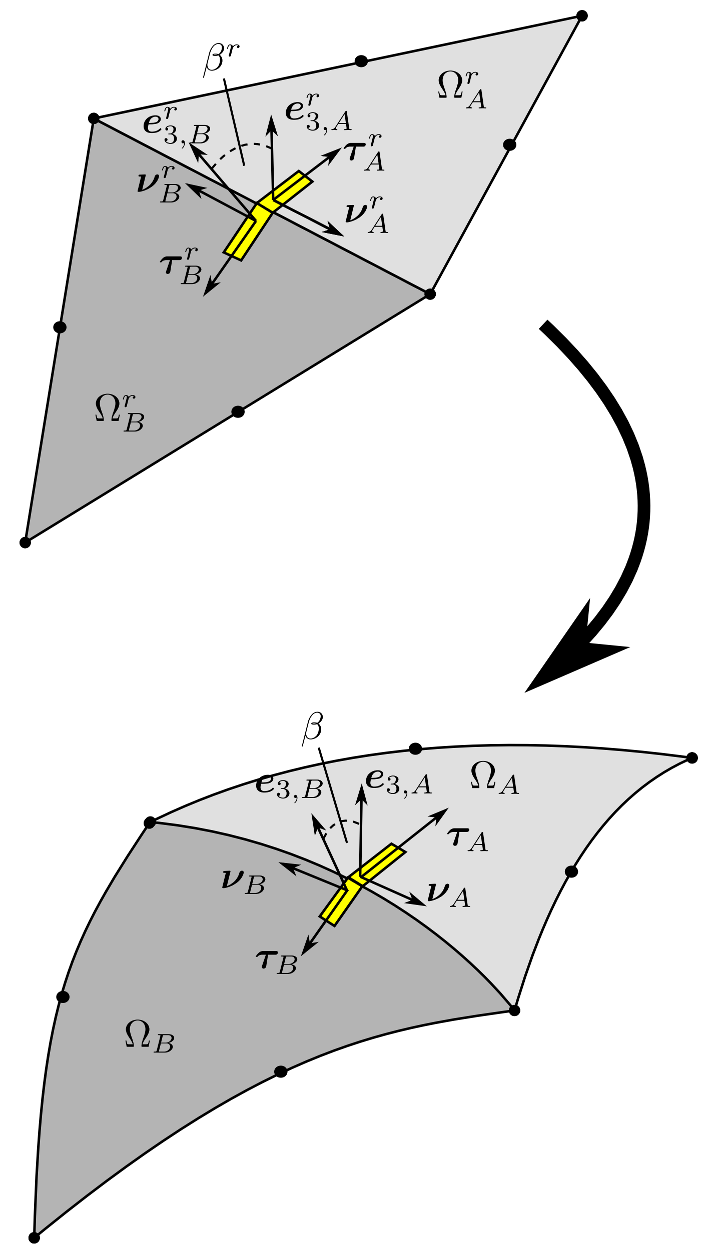
The C^1-continuity is asymptotically satisfied if β does not change during the motion $\rightarrow \beta - \beta^r = 0$. This is enforced, using a penalty approach, by

$$\Pi^{\text{pen}} = \int_{\Gamma^r} \frac{1}{2} k \left(\sin\beta - \sin\beta^r\right)^2 d\Gamma^r,$$

with $\sin\beta^{(r)} = (\boldsymbol{e}_{3,B}^{(r)} \times \boldsymbol{e}_{3,A}^{(r)}) \cdot \boldsymbol{\tau}_B^{(r)}$ and k as a penalty parameter.

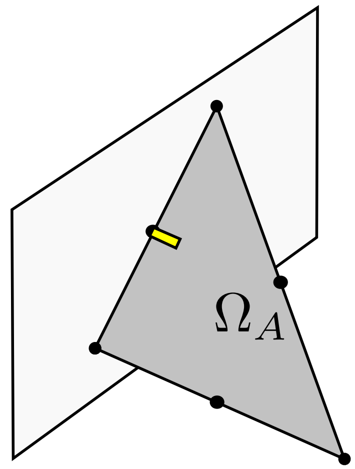
For this formulation no additional DOF is needed!

Alternatively the C^1-continuity could be enforced, using a Lagrange multiplier or the Augmented Lagrange method.



Enforcement of the C^1-Continuity

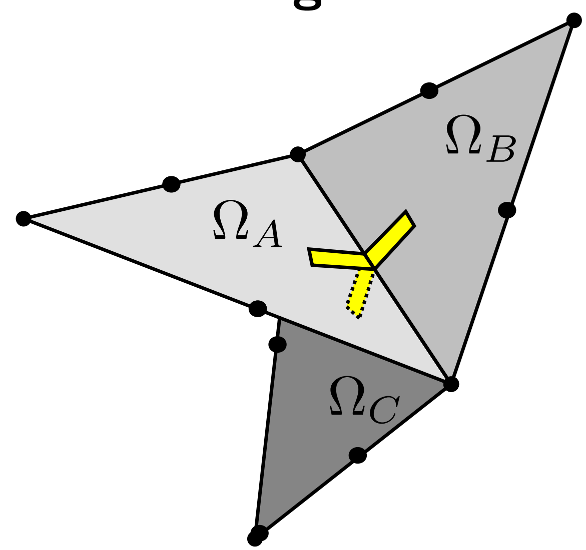
Clamped Edges



Clamping of free edges is enforced by minimization of

$$\Pi^{\text{pen,c}} = -\int_{\Gamma^r} \frac{1}{2}\, k\, ((\boldsymbol{e}^r_{3,A} \times \boldsymbol{e}_{3,A}) \cdot \boldsymbol{\tau}^r_A)^2 \, d\Gamma^r .$$

Branching shells



Multiple branched shells are adopted by minimization of

$$\Pi^{\text{pen,b}} = \int_{\Gamma^r} \frac{1}{2}\, k\, (\sin\beta_{AB} - \sin\beta^r_{AB})^2 \, d\Gamma^r + \int_{\Gamma^r} \frac{1}{2}\, k\, (\sin\beta_{AC} - \sin\beta^r_{AC})^2 \, d\Gamma^r .$$

Pinched Cylinder with rigid ends

$\psi = \frac{1}{4}\lambda((I_3 - 1) - \ln I_3) + \frac{1}{2}\mu(I_1 - 3 - \ln I_3)$

Geometrical Data: $r = 200,\ l = 400,\ h = 1$

Material Data: $E = 3 \cdot 10^4,\ \nu = 0.3$

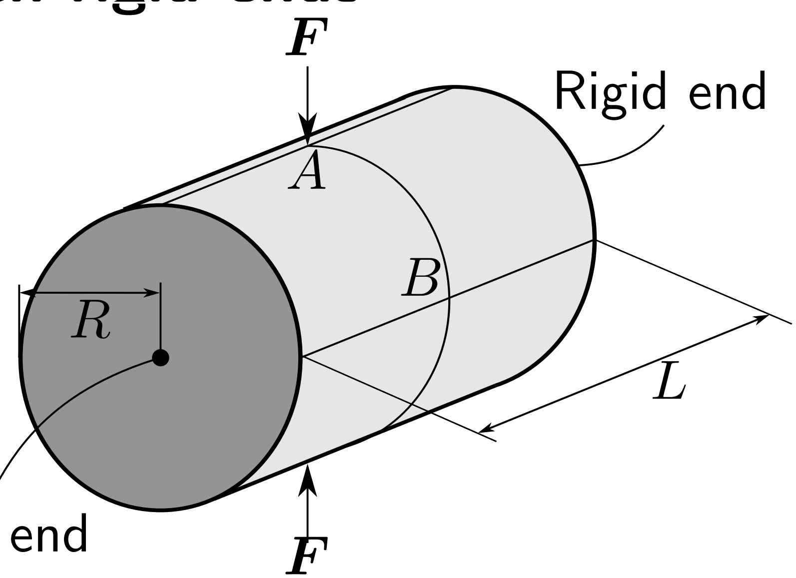
Penalty Parameter: $k = \dfrac{E\ h^3}{12(1-\nu^2)}$

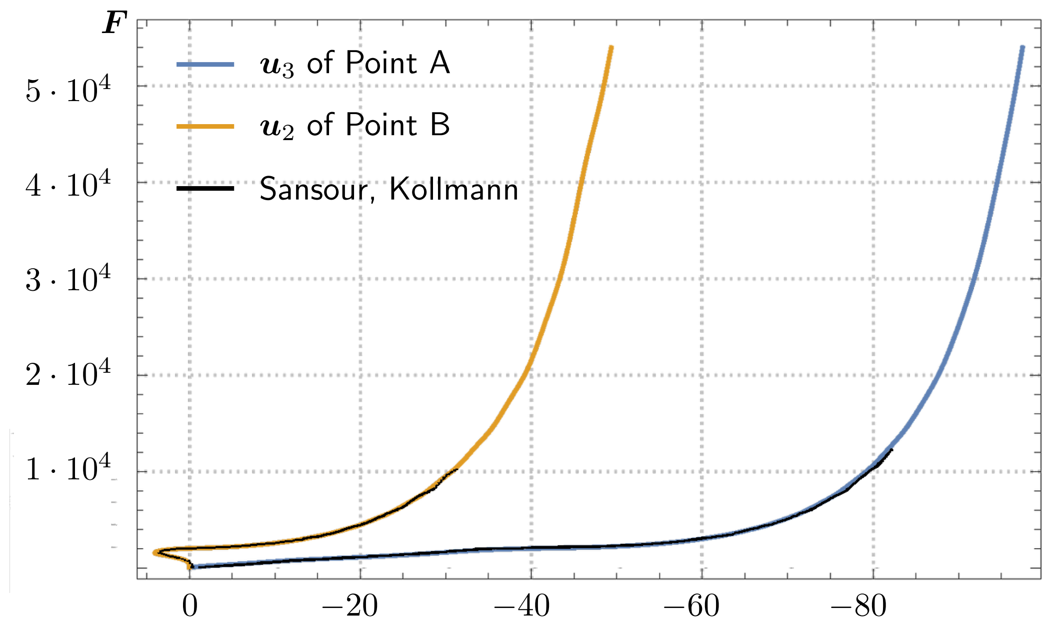
Boundary Conditions:

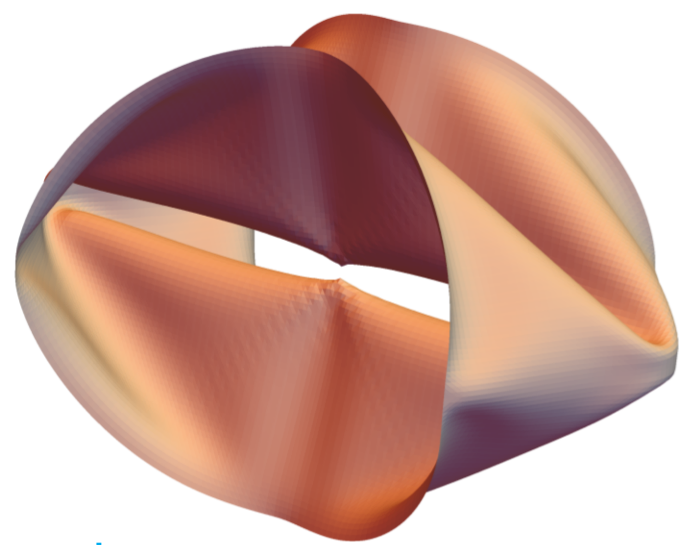
$u_2(x_1 = 0) = 0,\ u_3(x_1 = 0) = 0$

$u_2(x_1 = l\,) = 0,\ u_3(x_1 = l\,) = 0$

$F = 5.4 \cdot 10^4$







movie

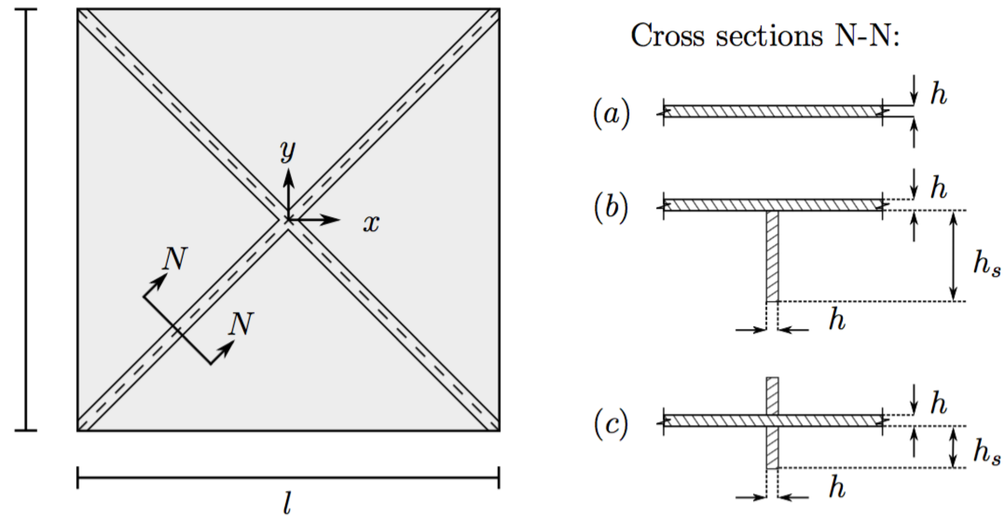
Plate with stiffeners

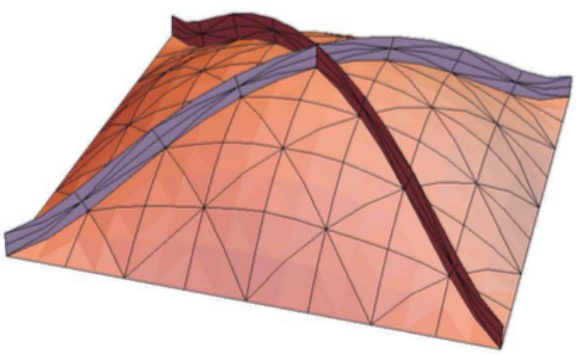
Geometrical Data: $l = 25.4$, $h = 0.254$

Stiffener: (b)$h_s = 1.27$, (c)$h_s = 0.508$

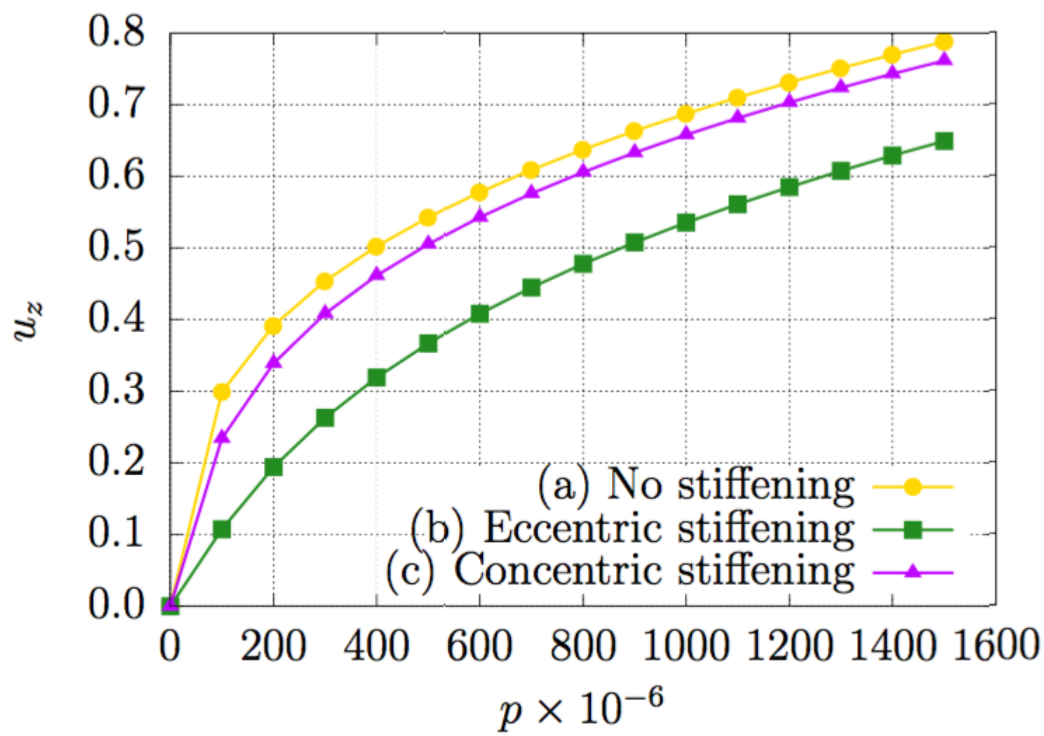
Material Data: $E = 117.25$, $\nu = 0.3$

Penalty Parameter: $k = \dfrac{E\ h^3}{12(1-\nu^2)}$





Deformations scaled by factor 10.



Dynamic reversion of clamped dome

Geometrical Data: $r = 0.05$, $h = 10^{-3}$

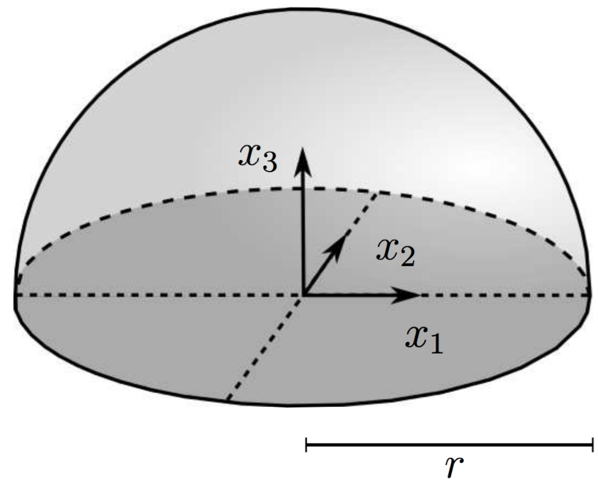
Material Data: $E = 10^5$, $\nu = 0.499$, $\rho = 1000$

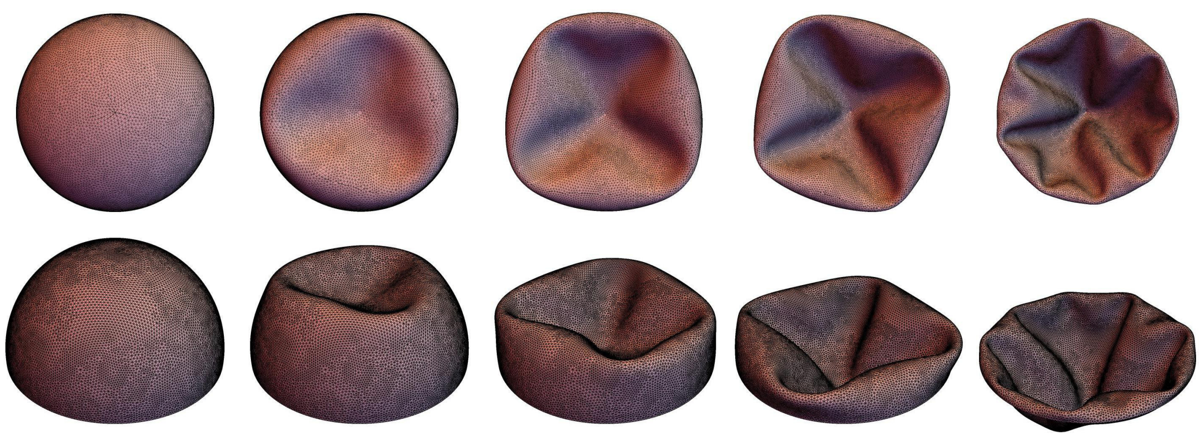
Penalty Parameter: $k = \dfrac{E\,h^3}{12(1-\nu^2)}$

Newmark Parameter: $\beta = 0.3025$, $\gamma = 0.6$

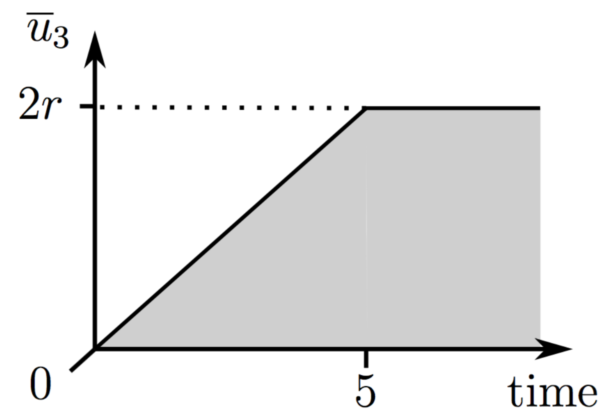
Boundary Conditions: $\boldsymbol{u}(\boldsymbol{x}_3 = 0) = \boldsymbol{0}$,

$$u_3(\boldsymbol{x} = (0,0,r)) = -2r$$





Time: 0.00 Time: 1.25 Time: 2.50 Time: 3.75 Time: 5.00



movie

Thank you for your attention

Acknowledgement: DFG Grants SCHR 570/23-2; 570/34-1;

SPP 1748 **Reliable Simulation Techniques in Solid Mechanics. Development of Non-standard Discretization Methods, Mechanical and Mathematical Analysis**

Jože Korelc - For the deployment of AceGen and AceFEM

Korelc J., Automatic generation of finite-element code by simultaneous optimization of expressions, Theoretical Computer Science, 1997, 187:231–248

Korelc J., Multi-language and Multi-environment Generation of Nonlinear Finite Element Codes, Engineering with Computers, 2002, 18:312–327

Least-squares functional for finite strain elasto-plasticity

First-order system, based on the multiplicative split of $\boldsymbol{F} = \boldsymbol{F}^e \boldsymbol{F}^p$,

$\boldsymbol{b}^e = \boldsymbol{F} \boldsymbol{C}^{p\,-1} \boldsymbol{F}^T$, $\psi(\boldsymbol{b}^e) = \frac{\Lambda}{4} \det \boldsymbol{b}^e + \frac{\mu}{2} \operatorname{tr} \boldsymbol{b}^e - (\frac{\Lambda}{2} + \mu) \ln \sqrt{\det \boldsymbol{b}^e}$:

$$\mathcal{F}(\boldsymbol{P}, \boldsymbol{u}) = \frac{1}{2} \Big(\|\omega_1 (\operatorname{Div} \boldsymbol{P} + \boldsymbol{f})\|_0^2 + \|\omega_2 (\boldsymbol{P}\boldsymbol{F}^T - 2 \frac{\partial \psi(\boldsymbol{b}^e)}{\partial \boldsymbol{b}^e} \boldsymbol{b}^e)\|_0^2 + \|\omega_3 (\boldsymbol{P}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{P}^T)\|_0^2 \Big) .$$

Principle of max. Dissipation; v. Mises criterion $\Phi = \| \operatorname{dev} \boldsymbol{\tau} \| + \sqrt{\frac{2}{3}} (y_0 + \beta(\alpha)) \leq 0$.

$$\mathcal{L}(\boldsymbol{\tau}, \beta, \gamma) = -\mathcal{D}_{int}(\boldsymbol{\tau}, \beta) + \gamma \, \Phi(\boldsymbol{\tau}, \beta) \rightarrow stat. \quad \text{with} \quad \gamma \geq 0$$

$$\partial_{\boldsymbol{\tau}} \mathcal{L} \Rightarrow \quad \frac{1}{2} \pounds(\boldsymbol{b}^e) \boldsymbol{b}^{e\,-1} = -\gamma \, \boldsymbol{n} \quad \Rightarrow \quad \boldsymbol{C}_{n+1}^{p\,-1} = \boldsymbol{F}_{n+1}^{-1} \exp[-2\lambda \boldsymbol{n}] \boldsymbol{F}_{n+1} \boldsymbol{C}_n^{p\,-1} ,$$

$$\partial_{\beta} \mathcal{L} \Rightarrow \quad \dot{\alpha} = \gamma \sqrt{\frac{2}{3}} \quad \Rightarrow \quad \alpha_{n+1} = \alpha_n + \sqrt{\frac{2}{3}} \lambda ,$$

fulfilling the yield criterion at time t_{n+1} yields $\lambda = \Delta t \gamma = \frac{3\Phi^{trial}}{2h}$.

Hill [1950], Weber & Anand [1990], Eterovic & Bathe [1990], Lubliner [1990]
Simo [1988a,1988b,1992,1998], Miehe & Stein [1992], ...

Cook's membrane problem for finite strain plasticity

Left side: $\boldsymbol{u} = (0,0,0)^T$

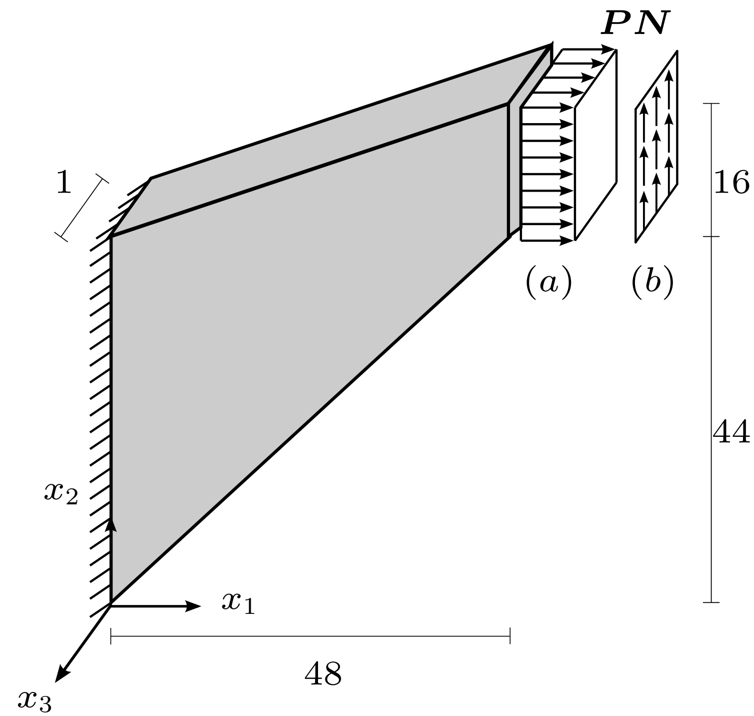
(a) Right face: $\boldsymbol{PN} = (4.5, 0, 0)^T$

(b) Right face: $\boldsymbol{PN} = (0, 2.5, 0)^T$

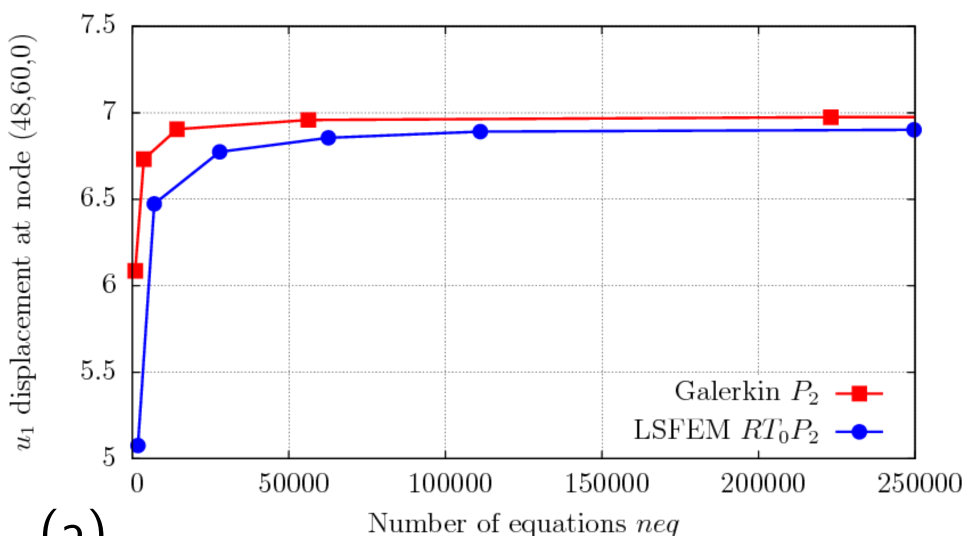
$E = 2069,\ \nu = 0.29,$

$y_0 = 4.5,\ h = 15$

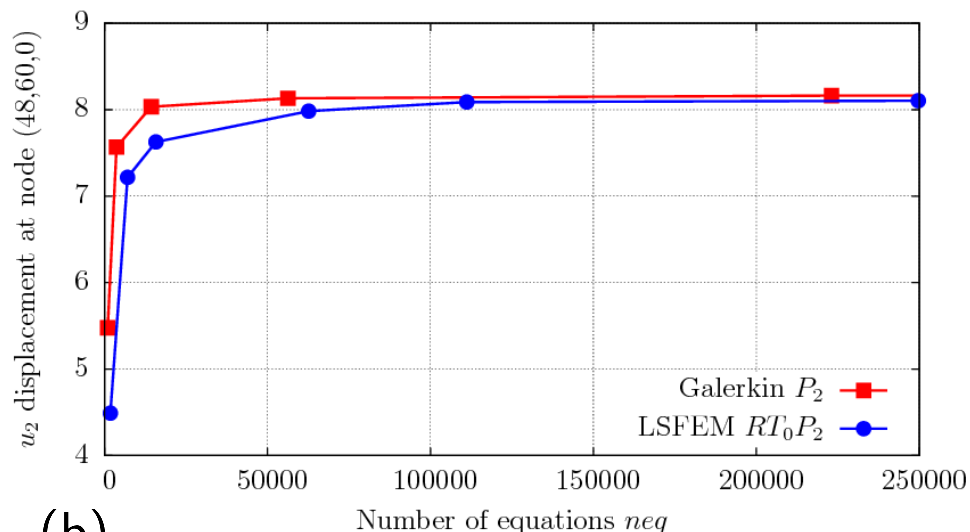
$\omega_1 = 1$, $\omega_2 = 1/\mu$ and $\omega_3 = 10/\mu$



Convergence studies for load cases (a) and (b):



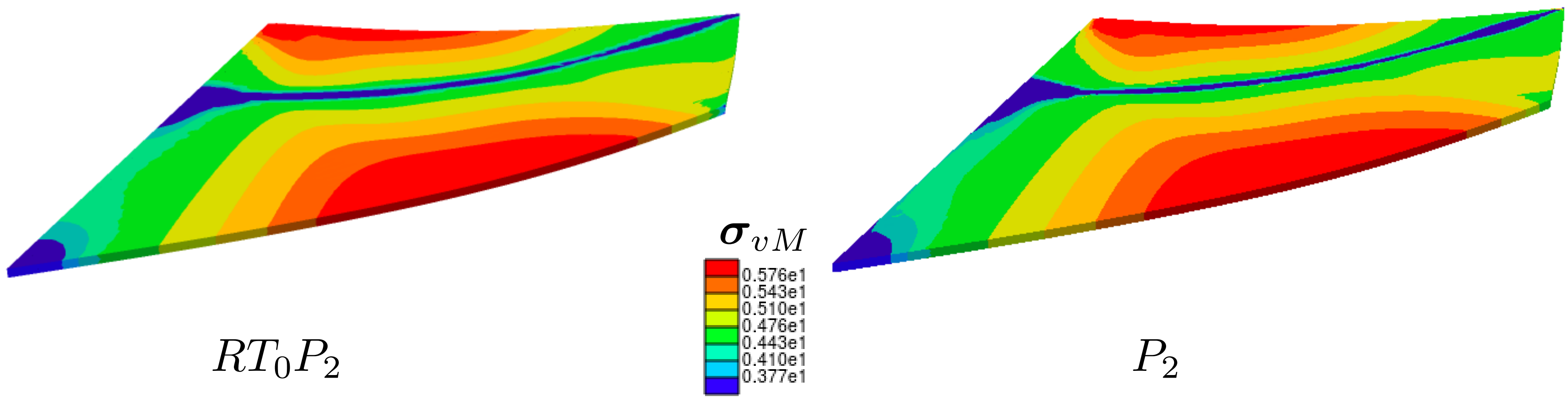
(a)



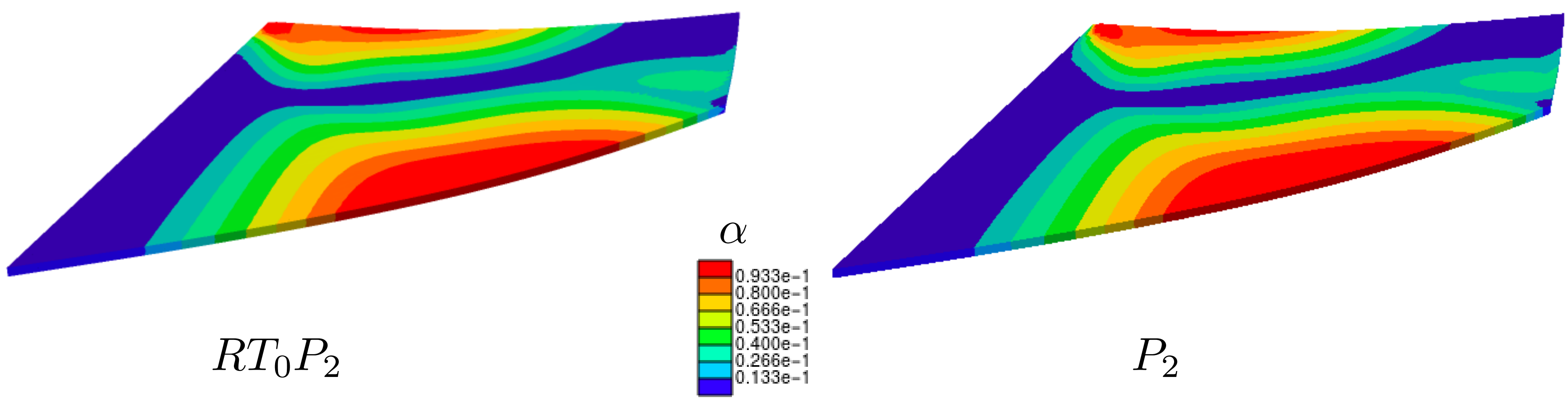
(b)

Cook's membrane problem for finite strain plasticity

Plot of von Mises stress $\boldsymbol{\sigma}_{vM}$ for $\boldsymbol{PN} = (0, 2.5, 0)^T$:



Plot of equivalent plastic strains α for $\boldsymbol{PN} = (0, 2.5, 0)^T$:



Hyperbolic shell (Balzani et al. [2008])

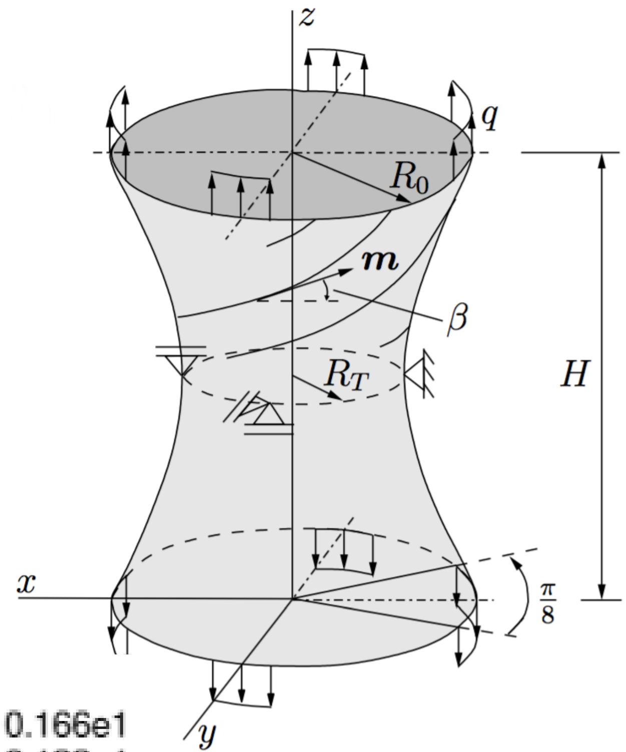
$$\psi = c_1\left(\frac{I_1}{I_3^{1/3}} - 3\right) + \epsilon_1(I_3^{\epsilon_2} + I_3^{-\epsilon_2} - 2) + \alpha_1\langle I_1 I_4 - I_5 - 2\rangle^{\alpha_2}$$

Geometrical Data: $R_0 = 5,\ H = 12,\ h = 0.05$

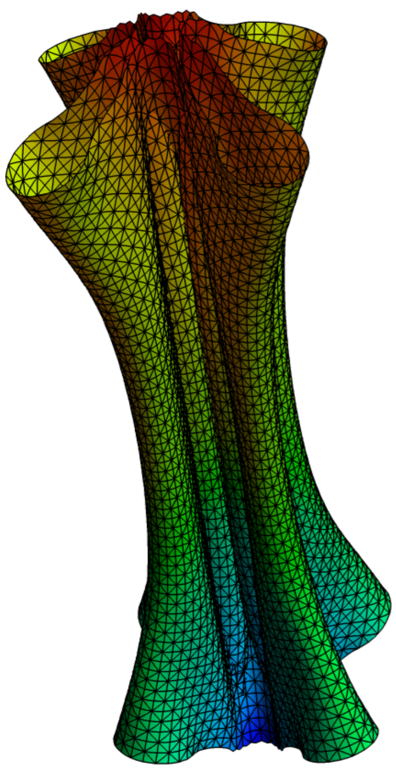
Material Data: $C_1 = 100,\ \epsilon_1 = 2000,\ \epsilon_2 = 10$

IF Tr. Iso.: $\alpha_1 = 1000,\ \alpha_2 = 2.3$

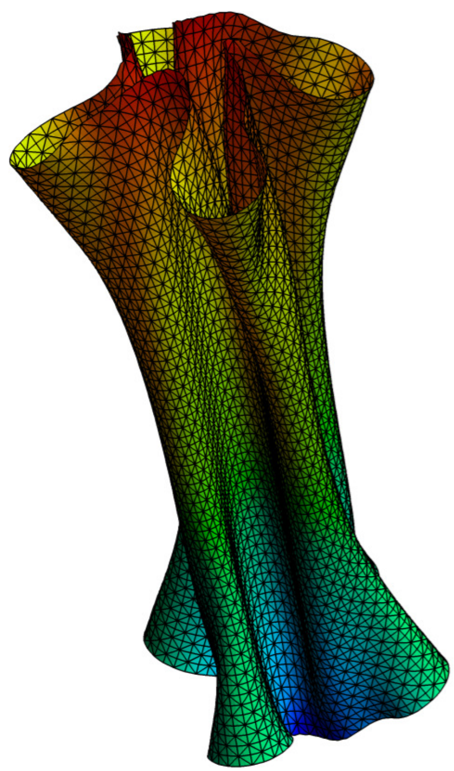
Penalty Parameter: $k = 10^4$

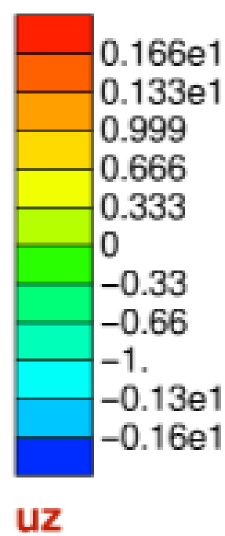


Isotropic



Transversal Isotropic





Algorithmic Treatment

ELEMENT LOOP

(1) Update displacements and stresses (Newton iteration k+1)

$$\boldsymbol{d} = \boldsymbol{d}_n^{(k)} + \Delta\boldsymbol{d},\ \boldsymbol{\beta} = \boldsymbol{\beta}_n^{(k)} + \Delta\boldsymbol{\beta}$$

INTEGRATION LOOP

(2) Compute stresses $\boldsymbol{S}$ and Green-Lagrange strain tensor $\boldsymbol{E}$ in each Gauss Point:

$$\underline{\boldsymbol{S}} = \underline{\boldsymbol{L}}\,\underline{\boldsymbol{\beta}},\ \underline{\boldsymbol{E}} = \underline{\boldsymbol{B}}\,\underline{\boldsymbol{d}},$$

Read from history: $\boldsymbol{E}^{\text{cons}}$

CONSTITUTIVE LOOP

(3) Compute residuum: $\boldsymbol{r}(\boldsymbol{E}^{\text{cons}}) = \boldsymbol{S} - \boldsymbol{S}^{\text{cons}}$

with $\boldsymbol{S}^{\text{cons}} = \partial_{\boldsymbol{E}^{\text{cons}}}\psi(\boldsymbol{E}^{\text{cons}})$

(4) Update: $\boldsymbol{E}^{\text{cons}} = \boldsymbol{E}^{\text{cons}} + \mathbb{D} : \boldsymbol{r}(\boldsymbol{E}^{\text{cons}})$

with $\mathbb{D} = (\partial_{\boldsymbol{E}^{\text{cons}}}\boldsymbol{S}^{\text{cons}})^{-1}$

(5) Check convergence

IF $||\mathbb{D} : \boldsymbol{r}(\boldsymbol{E}^{\text{cons}})||^2 \leq tol$

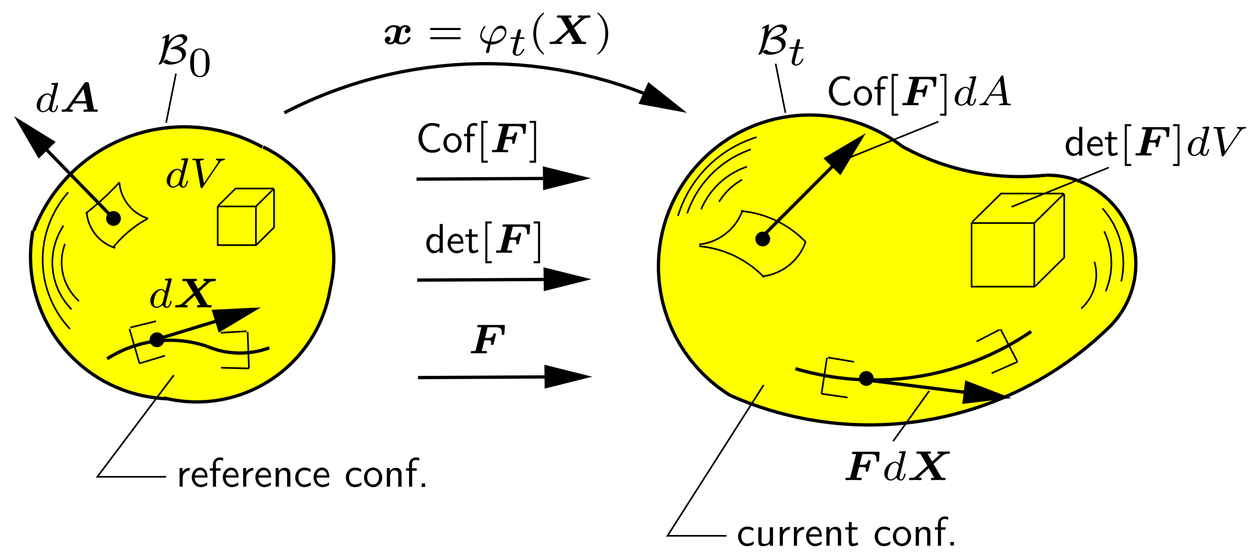
THEN Update History $\boldsymbol{E}^{\text{cons}}$ and exit CONSTITUTIVE LOOP

(6) Check divergence

IF $n_{\text{iter}} > n_{\text{tol}}$ THEN Stop Calculation

(7) Determine and export element stiffness and rhs-vector

Deformation of Line-, Area- and Volumeelement



Deformation of infinitesimal line element $\quad d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{X}$

Deformation of vectorial area element $d\boldsymbol{A}$:

$$d\boldsymbol{a} = (\boldsymbol{F}\,d\overset{1}{\boldsymbol{X}}) \times (\boldsymbol{F}\,d\overset{2}{\boldsymbol{X}}) = \mathrm{Cof}\,\boldsymbol{F}(d\overset{1}{\boldsymbol{X}} \times d\overset{2}{\boldsymbol{X}}) = \mathrm{Cof}[\boldsymbol{F}]\,d\boldsymbol{A}$$

Deformation of infinitesimal volume element

$$dv = d\boldsymbol{a} \cdot \boldsymbol{F}\,d\overset{3}{\boldsymbol{X}} = \mathrm{Cof}[\boldsymbol{F}]\,d\boldsymbol{A} \cdot \boldsymbol{F}d\overset{3}{\boldsymbol{X}} = J\,d\boldsymbol{A} \cdot d\overset{3}{\boldsymbol{X}} = J\,dV$$

Summary of Balance Equations in the Material Setting

Conservation of mass (densities $\rho_0 \in \mathcal{B}_0,\ \rho \in \mathcal{B}_t$)

$$\rho_0 = \rho\, J$$

Balance of linear momentum (body force $\rho_0 \boldsymbol{b}$)

$$\mathrm{Div}\, \boldsymbol{P} + \rho_0 \boldsymbol{b} = \rho_0 \ddot{\boldsymbol{x}}$$

Balance of moment of momentum

$$\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T$$

Balance of energy (internal energy e, heat flux vector $\boldsymbol{q}_0$ on $\partial\mathcal{B}_0$)

$$\rho_0\, \dot{e} = \boldsymbol{P} \cdot \dot{\boldsymbol{F}} - \mathrm{Div}\, \boldsymbol{q}_0 + \rho_0\, r$$

Clausius-Duhem inequality (free energy ψ, entropy η, absolute temperature Θ)

$$\boldsymbol{P} \cdot \dot{\boldsymbol{F}} - \rho_0 \left(\dot{\psi} + \dot{\Theta}\eta\right) - \frac{1}{\Theta}\boldsymbol{q}_0 \cdot \mathrm{Grad}\, \Theta \geq 0$$

Definition of Hyperelasticity

A material is termed hyperelastic if the existence of a free-energy ψ is postulated.

Evaluating the Clausius-Duhem relation, neglecting thermal effects yields

$$\boldsymbol{P} \cdot \dot{\boldsymbol{F}} - \rho_0 \dot{\psi}(\boldsymbol{F}) = 0 \quad \rightarrow \quad \boldsymbol{P} = \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}}$$

Internal work during quasi-static process in time interval $[\mathrm{t}_0, \mathrm{t}_1]$ for homogeneous deformation depends only on the values of ψ at the initial and final placement:

$$\int_{t_0}^{t_1} \boldsymbol{P} \cdot \dot{\boldsymbol{F}} \, dt = \int_{t_0}^{t_1} \rho_0 \frac{\partial \psi}{\partial \boldsymbol{F}} \cdot \dot{\boldsymbol{F}} \, dt = \rho_0 \int_{t_0}^{t_1} \dot{\psi} \, dt = \rho_0 \left(\psi(\boldsymbol{F}_1) - \psi(\boldsymbol{F}_0) \right)$$

Internal work during closed process is zero, i.e.

$$\rho_0 \int_{t_0}^{t_1} \dot{\psi} \, dt + \rho_0 \int_{t_1}^{t_2} \dot{\psi} \, dt = \rho_0 \left(\psi(\boldsymbol{F}_1) - \psi(\boldsymbol{F}_0) \right) + \rho_0 \left(\psi(\boldsymbol{F}_0) - \psi(\boldsymbol{F}_1) \right) = 0$$

where $\boldsymbol{F}_0 = \boldsymbol{F}(t_0)\,, \quad \boldsymbol{F}_1 = \boldsymbol{F}(t_1)\,, \quad \boldsymbol{F}_2 = \boldsymbol{F}(t_2) = \boldsymbol{F}_0$